2007

ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round : March 31, 2007
Second Round: April 21, 2007 at The University of Alabama

ALGEBRA II WITH TRIGONOMETRY EXAM

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D, and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered, and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one of more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then:
  $m\angle A$ is the measure of angle A in degrees
- If A and B are points on a circle, then:
  $\overset{\frown}{AB}$ is the arc between A and B
  $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees

Editing by Zhijian Wu, The University of Alabama
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What You Can Do With A Mathematics Major

Occupational opportunities

Actuarial and Insurance	Government	Accountant
Computer & Information Sciences	Investment Analyst	Financial Planner
Researcher	Benefits Specialist	Mathematician
Demographers	Computer Programmer	Cartographer
Data Processor	Navigator	Meteorologist
Applications Programmer	Ecologist	Health
Systems Analyst	Biomedical Engineer	Bio-mathematician
Computer Applications Engineer	Operations Analyst	Operations Research
Control Systems Engineer	Control Systems Engineer	Systems Engineer
Numerical Analyst	Teaching	BusinessIndustry
Statistician	Engineering Analyst	Financial Analyst
Technical Writer	Homeland Security	Communications Engineer

Study in the field of mathematics offers an education with an emphasis on careful problem analysis, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, health and environmental fields, and the academic world require sophisticated mathematical techniques for their solution. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of Alabama offers undergraduate and graduates degrees in Mathematics. Please visit **www.ua.edu** and refer to the undergraduate and graduate programs for additional information.

Engineering Math Advancement Program

The University of Alabama is offering a summer program to build math skills for students entering engineering. The Engineering Math Advancement Program, E-MAP, is a five-week summer residence class that addresses math and engineering prerequisites for incoming engineering students. The program targets bright students who may not have retained the information learned in high school and provides and opportunity to hone technical abilities before entering college. The goal of E-MAP is to assisting entering freshmen in developing a solid background in calculus to succeed in engineering before they start at the University.

Classes are designed around Precalculus Algebra and Trigonometry and incorporate important learning principles to ensure that knowledge is retained and not just memorized. Students develop their skills through hands-on experiences, problem solving teaming exercises, and interaction with engineering professors and instructors through an interdisciplinary Living Laboratory program. Experiments allow students to use simple calculus in engineering applications. The program also involves introducing students to local practicing engineers through work on one or more community service engineering-related activities. E-MAP will reserve 33-40 percent of enrollment space for underrepresented groups. Financial assistance is available based on need. Please visit **www.emap.ua.edu** for additional information.

1. Find the product

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2008}\right).$$

(**A**) $\dfrac{1}{2008!}$ (**B**) $\dfrac{1}{2007}$ (**C**) $\dfrac{2007!}{2^{2008}}$ (**D**) $\boxed{\dfrac{1}{2008}}$ (**E**) none of these

2. What is the value of $256^{-(2^{-2})}$?

(**A**) $\dfrac{1}{16}$ (**B**) 16777216 (**C**) $\boxed{\dfrac{1}{4}}$ (**D**) 65536 (**E**) 4

3. Simplify $\dfrac{\dfrac{x^2}{y}+\dfrac{y^2}{x}}{y^2-xy+x^2}$.

(**A**) x^2-y^2 (**B**) $\dfrac{xy}{x-y}$ (**C**) $\boxed{\dfrac{x+y}{xy}}$ (**D**) $x-y$ (**E**) none of these

4. If $g(x)=1-x^2$ and $f(g(x))=\dfrac{1-x^2}{x^2}$, find $f\left(\dfrac{3}{4}\right)$.

(**A**) 1 (**B**) 2 (**C**) -1 (**D**) $\boxed{3}$ (**E**) 0

5. What is the value of $\log_{27} 3$?

(**A**) $\boxed{\dfrac{1}{3}}$ (**B**) 2 (**C**) $-\dfrac{1}{3}$ (**D**) 3 (**E**) none of these

6. Evaluate $\sin\left(\arcsin\dfrac{3}{5}+\arcsin\dfrac{8}{17}\right)$.

(**A**) $\dfrac{4}{5}$ (**B**) $\boxed{\dfrac{77}{85}}$ (**C**) $\dfrac{84}{85}$ (**D**) $\dfrac{91}{85}$ (**E**) $\dfrac{15}{17}$

7. If $f\left(\dfrac{1-x}{1+x}\right)=x$, then $f(x)$ is

(**A**) $\dfrac{1+x}{1-x}$ (**B**) $\boxed{\dfrac{1-x}{1+x}}$ (**C**) $\dfrac{x+1}{x-1}$ (**D**) $\dfrac{2x}{x+1}$ (**E**) none of these

8. If $3^a=4^b=36$, find $\dfrac{2}{a}+\dfrac{1}{b}$.

(**A**) -1 (**B**) 0 (**C**) $\boxed{1}$ (**D**) 2 (**E**) 3

9. Find the distance from the point $(3,2)$ to the line $y=3x+2$.

(**A**) $\dfrac{2}{\sqrt{3}}$ (**B**) 3 (**C**) $\dfrac{3}{\sqrt{5}}$ (**D**) 5 (**E**) $\boxed{\dfrac{9}{\sqrt{10}}}$

10. If $\log_x \sqrt{8} = \frac{1}{2}$, find x.

(**A**) 2 (**B**) 3 (**C**) 4 (**D**) $\boxed{8}$ (**E**) 9

11. Evaluate

$$\cos\frac{\pi}{5} + \cos\frac{2\pi}{5} + \cos\frac{3\pi}{5} + \cos\frac{4\pi}{5}.$$

(**A**) $-\frac{1}{2}$ (**B**) $\boxed{0}$ (**C**) $\frac{1}{2}$ (**D**) 1 (**E**) $\frac{\sqrt{3}}{2}$

12. The period of the function $\cos^4 x - \sin^4 x$ is

(**A**) $\frac{\pi}{4}$ (**B**) $\frac{\pi}{2}$ (**C**) $\frac{3\pi}{4}$ (**D**) $\boxed{\pi}$ (**E**) $\frac{3\pi}{2}$

13. Suppose $x, y, z > 0$, and $\ln x + \ln y + \ln z = 0$. Find

$$x^{\frac{1}{\ln x}+\frac{1}{\ln z}} \cdot y^{\frac{1}{\ln z}+\frac{1}{\ln x}} \cdot z^{\frac{1}{\ln x}+\frac{1}{\ln y}}.$$

(**A**) $\boxed{e^{-3}}$ (**B**) 1 (**C**) e (**D**) e^2 (**E**) e^3

14. Determine the radius of the circle given by $x^2 + y^2 - 6x + 8y = 56$.

(**A**) 3.5 (**B**) 6 (**C**) $5\sqrt{2}$ (**D**) $\boxed{9}$ (**E**) $4\sqrt{14}$

15. Find the distance between the point $x = -1$ and the midpoint of the domain of the function

$$f(x) = \sqrt{4 - \sqrt{2x+5}}.$$

(**A**) $\frac{5}{8}$ (**B**) $\boxed{\frac{5}{2}}$ (**C**) $\frac{1}{4}$ (**D**) $\frac{2}{3}$ (**E**) $\frac{2}{5}$

16. Given $0 < \alpha, \beta < 90^\circ$, $\cos\alpha = \frac{3}{5}$ and $\cos(\alpha+\beta) = -\frac{12}{13}$, find $\sin\beta$.

(**A**) $\frac{56}{65}$ (**B**) $\frac{11}{12}$ (**C**) $\boxed{\frac{63}{65}}$ (**D**) $\frac{4}{5}$ (**E**) none of these

17. If $a^2 - a - 1 = 0$, find the value of $a^8 + \frac{1}{a^8}$.

(**A**) 15 (**B**) 21 (**C**) 33 (**D**) 42 (**E**) $\boxed{47}$

18. Evaluate

$$(2+1)(2^2+1)(2^4+1)\cdots(2^{32}+1)+1.$$

(**A**) 2^8 (**B**) 2^{16} (**C**) 2^{32} (**D**) $\boxed{2^{64}}$ (**E**) none of these

19. Evaluate

$$\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ.$$

(**A**) 1 (**B**) $\frac{2}{3}$ (**C**) $\frac{1}{2}$ (**D**) $\boxed{\frac{1}{8}}$ (**E**) none of these

20. Find the range of the function $y = \dfrac{x^2 - x + 1}{2x^2 - 2x + 3}$.

(**A**) $\boxed{[\frac{3}{10}, \frac{1}{2})}$ (**B**) $[\frac{1}{3}, \frac{1}{2})$ (**C**) $(2, 3]$ (**D**) $[\frac{1}{2}, 2)$ (**E**) $[\frac{1}{2}, 3)$

21. If $\sin x + \sin^2 x = 1$, find the value of $\cos^2 x + \cos^4 x$.

(**A**) $\boxed{1}$ (**B**) $\dfrac{1}{3\sqrt{5}}$ (**C**) $\dfrac{1}{2}(3\sqrt{5} - 5)$ (**D**) -1 (**E**) $\dfrac{1}{2}$

22. Evaluate

$$\left(\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2007}\right)\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2006}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2007}\right)\left(\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2006}\right).$$

(**A**) $-\dfrac{1}{2007}$ (**B**) $\boxed{\dfrac{1}{2007}}$ (**C**) $\dfrac{2}{2007}$ (**D**) $\dfrac{3}{2007}$ (**E**) none of these

23. If $\tan\alpha$ and $\tan\beta$ are two solutions of $x^2 - px + q = 0$, $\cot\alpha$ and $\cot\beta$ are two solutions of $x^2 - rx + s = 0$, what is the value of rs in terms of p and q?

(**A**) pq (**B**) $\dfrac{1}{pq}$ (**C**) $\boxed{\dfrac{p}{q^2}}$ (**D**) $\dfrac{q}{p^2}$ (**E**) none of these

24. If $1 + x + x^2 + x^3 = 0$, what is the value of

$$1 + x + x^2 + x^3 + x^4 + ... + x^{2008}\,?$$

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 3 (**E**) 4

25. If $a < b < c$, find the minimum value of $y = |x - a| + |x - b| + |x - c|$.

(**A**) 0 (**B**) $a + c$ (**C**) c (**D**) a (**E**) $\boxed{c - a}$

26. Given $0 < x < \pi$ and $\sin x + \cos x = \dfrac{1}{5}$, find $\tan x$.

(**A**) $\boxed{-\dfrac{4}{3}}$ (**B**) $-\dfrac{3}{4}$ (**C**) $\dfrac{4}{3}$ (**D**) $\dfrac{3}{4}$ (**E**) $\dfrac{3}{5}$

27. If a, b and c are nonzero rational numbers, then what is the possible value of $\dfrac{|a|}{a} + \dfrac{|b|}{b} + \dfrac{|c|}{c}$?

(**A**) $3, 1$ (**B**) $-3, 1, -1$ (**C**) $3, 1, -1$ (**D**) $3, -3, 0, 1, -1$ (**E**) $\boxed{3, -3, 1, -1}$

28. Evaluate

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{(3n - 2) \cdot (3n + 1)}.$$

(**A**) $\boxed{\dfrac{n}{(3n + 1)}}$ (**B**) $\dfrac{n}{(3n - 1)}$ (**C**) $\dfrac{2n}{(3n + 1)}$ (**D**) $\dfrac{n}{(n + 1)}$ (**E**) $\dfrac{n}{(n - 1)}$

29. If equation $4k(x+2)-1=2x$ yields no solution to x, what is the value of k?

(**A**) -1 (**B**) 1 (**C**) $\boxed{\frac{1}{2}}$ (**D**) -3 (**E**) 2

30. Given $p>0$ and $m>n$, compare the values of m, n and $\frac{m+np}{1+p}$.

(**A**) $n<m<\frac{m+np}{1+p}$ (**B**) $\frac{m+np}{1+p}<n<m$

(**C**) $n<\frac{m+np}{1+p}<\frac{m}{2}$ (**D**) $\boxed{n<\frac{m+np}{1+p}<m}$ (**E**) none of these

31. If $x+y=1$ and $x^2+y^2=2$, find the value of x^7+y^7.

(**A**) 8 (**B**) $\frac{61}{8}$ (**C**) $\boxed{\frac{71}{8}}$ (**D**) 7 (**E**) 9

32. Given $\tan\alpha=2$, find $\cos^2\alpha-\sin^2\alpha$.

(**A**) $-\frac{4}{3}$ (**B**) $\boxed{-\frac{3}{5}}$ (**C**) $\frac{5}{3}$ (**D**) $\frac{3}{4}$ (**E**) $\frac{5}{4}$

33. Suppose a,b,c are nonnegative numbers, and $3a+2b+c=5$, $2a+b-3c=1$. Find the maximum value of $S=3a+b-7c$.

(**A**) $-\frac{5}{7}$ (**B**) $\boxed{-\frac{1}{11}}$ (**C**) $\frac{1}{11}$ (**D**) $\frac{5}{7}$ (**E**) 0

34. Suppose n is a natural number which makes the number $n^2-19n+91$ a complete square. How many such n's are there?

(**A**) none (**B**) 1 (**C**) $\boxed{2}$ (**D**) 3 (**E**) 4

35. Compare $A=\frac{2007^{2006}+1}{2007^{2007}+1}$ and $B=\frac{2007^{2007}+1}{2007^{2008}+1}$.

(**A**) $A=B$ (**B**) $\boxed{A>B}$ (**C**) $A<B$ (**D**) $A<1<B$ (**E**) none of these

36. Solve the inequality $|x-5|-|2x+3|\le 1$.

(**A**) $(-7,-3)$ (**B**) $[-7,\frac{1}{3})$ (**C**) $\boxed{(-\infty,-7]\text{ or }[\frac{1}{3},\infty)}$ (**D**) $(-7,\frac{1}{3})$ (**E**) none of these

37. If $i=\sqrt{-1}$, what is i^{2007}?

(**A**) 0 (**B**) 1 (**C**) -1 (**D**) i (**E**) $\boxed{-i}$

38. How many *distinct* roots does $p(x)=x^7+3x^6+5x^5+7x^4+7x^3+5x^2+3x+1$ have?

(**A**) $\boxed{3}$ (**B**) 4 (**C**) 5 (**D**) 6 (**E**) 7

39. The equation $3^{x^2} = 81^{2x-3}$ has *two* solutions. What is the sum of the solutions?

(**A**) -2 (**B**) 0 (**C**) 4 (**D**) $\boxed{8}$ (**E**) none of these

40. Solve for n:

$$5^n + 5^n + 5^n + 5^n + 5^n = (\sqrt{5})^{10}.$$

(**A**) 5 (**B**) $\boxed{4}$ (**C**) $\frac{1}{2}$ (**D**) $\frac{2}{5}$ (**E**) none of these

41. If $0° < A < 90°$ and $\sin A + \cos A = \frac{7}{12}$, then the angle A satisfies

(**A**) $0° < A < 45°$ (**B**) $A = 45°$ (**C**) $45° < A < 90°$ (**D**) $\boxed{90° < A < 180°}$ (**E**) none of these

42. How many solutions does the equation $|x - 2| = \sqrt{2x}$ have?

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 4 (**E**) infinitely many

43. Find the sum of all the solutions to the equation $6 + \sqrt{x+6} = x$.

(**A**) 4 (**B**) 7 (**C**) 13 (**D**) 11 (**E**) $\boxed{10}$

44. If r and s are the (real) roots of the equation $x^2 + 3x + k = 0$, find k so that $|r - s| = 2$.

(**A**) $\boxed{\frac{5}{4}}$ (**B**) $\frac{3}{4}$ (**C**) $\frac{3}{5}$ (**D**) $\frac{5}{3}$ (**E**) 4

45. Find the sum of all the solutions to the equation $2\log x - \log(2x - 75) = 2$.

(**A**) no solutions (**B**) 30 (**C**) 350 (**D**) 75 (**E**) $\boxed{200}$

46. Evaluate $\log_3 \sqrt{3\sqrt{3\sqrt{3}}}$.

(**A**) $\boxed{\frac{7}{8}}$ (**B**) 8 (**C**) $\frac{7}{3}$ (**D**) 1 (**E**) 3

47. The function $f(x) = \dfrac{1}{x^2 + 2x + c}$ will have no vertical asymptote when the constant c is which one of the following?

(**A**) -2 (**B**) -1 (**C**) 0 (**D**) 1 (**E**) $\boxed{2}$

48. Let $a = \log_{12} 27$. Write $\log_6 16$ in terms of a.

(**A**) $\dfrac{4(3+a)}{3-a}$ (**B**) $\dfrac{2(3-a)}{3+a}$ (**C**) $\dfrac{(3-a)}{3+a}$ (**D**) $\dfrac{3(3+a)}{3-a}$ (**E**) $\boxed{\dfrac{4(3-a)}{3+a}}$

49. Solve the inequality $x^{1+\log_{\frac{1}{2}} x} > \frac{1}{4}x$.

(**A**) $\frac{1}{2} < x < 2$ (**B**) $\boxed{2^{-\sqrt{2}} < x < 2^{\sqrt{2}}}$ (**C**) $\frac{1}{\sqrt{2}} < x < \sqrt{2}$

(**D**) $\sqrt{2}^{-\sqrt{2}} < x < \sqrt{2}^{\sqrt{2}}$ (**E**) none of these

50. Solve $\sin x \leq \frac{1}{2}$, where $-\frac{\pi}{2} < x \leq \frac{\pi}{2}$.

(**A**) $\boxed{(-\frac{\pi}{2}, \frac{\pi}{6}]}$ (**B**) $(-\frac{\pi}{6}, \frac{\pi}{6}]$ (**C**) $(-\frac{\pi}{2}, \frac{\pi}{6})$ (**D**) $[0, \frac{\pi}{6}]$ (**E**) $(-\frac{\pi}{6}, \frac{\pi}{6})$