2008

A L A B A M A

STATEWIDE MATHEMATICS CONTEST

First Round : March 29, 2008
Second Round: April 26, 2008 at The University of Alabama

ALGEBRA II WITH TRIGONOMETRY EXAM

Construction of this test directed
by
Zhijian Wu, The University of Alabama

INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D, and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered, and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one of more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then:
  $m\angle A$ is the measure of angle A in degrees
- If A and B are points on a circle, then:
  $\widehat{AB}$ is the arc between A and B
  $m\,\widehat{AB}$ is the measure of $\widehat{AB}$ in degrees

Editing by Zhijian Wu, The University of Alabama
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What You Can Do With A Mathematics Major

Occupational opportunities

Actuarial and Insurance	Government	Accountant
Computer & Information Sciences	Investment Analyst	Financial Planner
Researcher	Benefits Specialist	Mathematician
Demographers	Computer Programmer	Cartographer
Data Processor	Navigator	Meteorologist
Applications Programmer	Ecologist	Health
Systems Analyst	Biomedical Engineer	Bio-mathematician
Computer Applications Engineer	Operations Analyst	Operations Research
Control Systems Engineer	Control Systems Engineer	Systems Engineer
Numerical Analyst	Teaching	BusinessIndustry
Statistician	Engineering Analyst	Financial Analyst
Technical Writer	Homeland Security	Communications Engineer

Study in the field of mathematics offers an education with an emphasis on careful problem analysis, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, health and environmental fields, and the academic world require sophisticated mathematical techniques for their solution. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of Alabama offers undergraduate and graduates degrees in Mathematics. Please visit **www.ua.edu** and refer to the undergraduate and graduate programs for additional information.

Engineering Math Advancement Program

The University of Alabama is offering a new summer program to build math skills for students entering engineering. The Engineering Math Advancement Program (EMAP) is a summer residence class that addresses math and engineering prerequisites for incoming engineering students. The program targets bright students who may not have retained the information learned in high school and provides and opportunity to hone technical abilities before entering college. The goal of E-MAP is to assisting entering freshmen in developing a solid background in calculus to succeed in engineering before they start at the University.

Classes are designed around Precalculus Algebra and Trigonometry and incorporate important learning principles to ensure that knowledge is retained and not just memorized. Students develop their skills through hands-on experiences, problem solving teaming exercises, and interaction with engineering professors and instructors through an interdisciplinary Living Laboratory program. Experiments allow students to use simple calculus in engineering applications. The program also involves introducing students to local practicing engineers through work on one or more community service engineering-related activities. E-MAP will reserve 33-40 percent of enrollment space for underrepresented groups. Financial assistance is available based on need. Please visit **emap.ua.edu** for additional information.

1. Determine the last digit of 3^{3^3} .

(**A**) 1 (**B**) 3 (**C**) $\boxed{7}$ (**D**) 9 (**E**) none of these

2. If $a \otimes b = ab + 1$ and $a \oplus b = a + b$, find the value of $4 \otimes [(6 \oplus 8) \oplus (3 \otimes 5)]$.

(**A**) 34 (**B**) 76 (**C**) 116 (**D**) 120 (**E**) $\boxed{121}$

3. If $\log ab + \log bc + \log ac = 10$ for positive a, b and c, what is the value of $\log abc$?

(**A**) $\frac{1}{2}$ (**B**) 1 (**C**) $\boxed{5}$ (**D**) 10 (**E**) 100

4. If $A = p(1 + r)^t(1 + s)^u$, where p, r, s, t, u are positive quantities, then $\frac{ln(A/p)}{ln(1+r)ln(1+s)}$ is equal to

(**A**) $(1 + r)t + (1 + s)u$ (**B**) $(1 + s)t + (1 + r)u$

(**C**) $st + ru$ (**D**) $\frac{t}{ln(1+r)} + \frac{u}{ln(1+s)}$ (**E**) $\boxed{\frac{t}{ln(1+s)} + \frac{u}{ln(1+r)}}$

5. How many real number solutions are there for the equation $(x^2 + 4x + 5)^{x^2+1} = 1$?

(**A**) 0 (**B**) $\boxed{1}$ (**C**) 2 (**D**) 3 (**E**) 4

6. Solve $\cos 2\theta = \sin\theta$, if θ is in $[0, 2\pi)$.

(**A**) $\frac{\pi}{2}, \frac{3}{2}\pi$ (**B**) $\frac{\pi}{3}, \frac{2}{3}\pi, \frac{3}{2}\pi$ (**C**) $\frac{\pi}{6}, \frac{5}{6}\pi, \pi$ (**D**) $\boxed{\frac{\pi}{6}, \frac{5}{6}\pi, \frac{3}{2}\pi}$ (**E**) $\frac{\pi}{4}, \frac{3}{4}\pi, \frac{3}{2}\pi$

7. Find all values of $2x - 5$ if $5 + \sqrt{x + 7} = x$.

(**A**) -1 (**B**) 1 (**C**) $\boxed{13}$ (**D**) 15 (**E**) 20

8. The difference between the largest and smallest roots of $30x^3 - 31x^2 - 66x + 72 = 0$ is

(**A**) $-\frac{3}{2}$ (**B**) $\frac{6}{5}$ (**C**) $\frac{4}{3}$ (**D**) $\boxed{\frac{17}{6}}$ (**E**) $\frac{27}{10}$

9. If $x + 2y + 3z = 6$, $2x - 3y + 2z = 14$ and $3x + y - z = -2$, then $x + y + z$ is

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 3 (**E**) 4

10. The number of common points shared by the graphs of $|x| + |y| = 2$ and $x^2 - y = 2$ are

(**A**) 0 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) $\boxed{5}$

11. Jack had an average score of 85 on his first eight quizzes, and an average score of 81 on his first nine quizzes. What score did he receive on his *ninth* quiz ?

(**A**) $\boxed{49}$ (**B**) 51 (**C**) 53 (**D**) 55 (**E**) 57

12. Suppose that $f(a) = a^a$ and $g(a) = a^{2a}$. Which of the functions below is equal to $f(g(a))$?

(**A**) a^{3a} (**B**) $a^{a^{2a}}$ (**C**) $a^{2a^{2a}}$ (**D**) $a^{2a^{3a}}$ (**E**) $\boxed{a^{2a^{2a+1}}}$

13. What is the value of the product $\sin\frac{\pi}{32}\cos\frac{\pi}{32}\cos\frac{\pi}{16}\cos\frac{\pi}{8}\cos\frac{\pi}{4}$?

(**A**) $\frac{1}{2}$ (**B**) $\frac{1}{4}$ (**C**) $\frac{1}{8}$ (**D**) $\boxed{\frac{1}{16}}$ (**E**) $\frac{1}{32}$

14. Suppose real numbers x and y satisfy $x^2+9y^2-4x+6y+4=0$. What is the maximum value of $4x-9y$?

(**A**) 15 (**B**) $9\sqrt{3}$ (**C**) $\boxed{16}$ (**D**) $12\sqrt{2}$ (**E**) 18

15. Find the value of the sum:

$$\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\frac{5}{3!+4!+5!}+\cdots+\frac{2008}{2006!+2007!+2008!}.$$

(**A**) $\frac{2008!+2}{2\cdot 2008!}$ (**B**) $\frac{2008!+1}{2\cdot 2008!}$ (**C**) $\frac{2007}{2008}$ (**D**) $\boxed{\frac{2008!-2}{2\cdot 2008!}}$ (**E**) $\frac{2008!-2}{2008!}$

16. If $x+y=0$ and $x\neq 0$, then what is the value of $\frac{x^{2008}}{y^{2008}}$?

(**A**) 2008 (**B**) -2008 (**C**) $\boxed{1}$ (**D**) -1 (**E**) 0

17. Fresh grapes contains 80% water by weight, whereas dried grapes contains 15% water by weight. How many pounds of dried grapes can be obtained from 34 pounds of fresh grapes?

(**A**) $\boxed{8}$ (**B**) 9 (**C**) 10 (**D**) 11 (**E**) 12

18. Which value of θ listed below leads to

$$2^{\sin\theta}>1 \quad \text{and} \quad 3^{\cos\theta}<1\,?$$

(**A**) 70° (**B**) $\boxed{140°}$ (**C**) 210° (**D**) 280° (**E**) 350°

19. Define a sequence by $b_1=2$ and $b_{n+1}=\frac{1+b_n}{1-b_n}$ for $n>0$. What is the value of b_{2008}?

(**A**) 2008 (**B**) 2 (**C**) -3 (**D**) $-\frac{1}{2}$ (**E**) $\boxed{\frac{1}{3}}$

20. Solve for a in terms of b: $\frac{a}{b}=a-b$.

(**A**) $a=\frac{b}{b-1}$ (**B**) $\boxed{a=\frac{b^2}{b-1}}$ (**C**) $a=\frac{b}{b+1}$ (**D**) $a=\frac{b^2}{1-b}$ (**E**) none of these

21. How many polynomials are there of the form x^3-8x^2+cx+d such that c and d are real numbers and the three roots of the polynomial are distinct positive integers?

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 3 (**E**) 4

22. Let $f(x)$ be a polynomial of degree 2006 satisfying

$$f(k) = \frac{1}{k}, \quad 1 \leq k \leq 2007\,.$$

What is the value of $f(2008)$?

(**A**) 0 (**B**) $\frac{1}{2008}$ (**C**) $\boxed{\frac{2}{2008}}$ (**D**) $\frac{3}{2008}$ (**E**) $\frac{4}{2008}$

23. The difference of two positive numbers is 2, and the product of these two numbers is 17. What is their sum ?

(**A**) $\boxed{6\sqrt{2}}$ (**B**) $3\sqrt{2}-2$ (**C**) $6\sqrt{2}-2$ (**D**) $3\sqrt{2}$ (**E**) $3\sqrt{2}+2$

24. One solution of $x^3+5x^2-2x-4=0$ is $x=1$. Which of the following is another solution ?

(**A**) $-1+\sqrt{7}$ (**B**) $\boxed{-3+\sqrt{5}}$ (**C**) $-2+\sqrt{5}$ (**D**) $-3+\sqrt{3}$ (**E**) $-5+\sqrt{2}$

25. Mike and Dave play a game in which each independently throws a dart at a target. Mike hits the target with probability 0.6, while Dave hits the target with probability 0.3. Mike wins the game if he hits the target and Dave misses. Dave wins the game if he hits the target and Mike misses. Otherwise the game is a tie. What is the probability that the game is a tie ?

(**A**) 0.45 (**B**) $\boxed{0.46}$ (**C**) 0.47 (**D**) 0.48 (**E**) 0.49

26. The quadratic polynomial $p(x)$ has the following properties: $p(x) \geq 0$ for all real numbers x, $p(1)=0$ and $p(2)=2$. What is the value of $p(0)+p(3)$?

(**A**) 9 (**B**) $\boxed{10}$ (**C**) 11 (**D**) 12 (**E**) 13

27. If $f(x)$ satisfies $2f(x)+f(1-x)=x^2$ for all x, then $f(x)=$

(**A**) $\frac{x^2-3x+1}{2}$ (**B**) $\frac{x^2+8x-3}{9}$ (**C**) $\frac{4x^2+3x-2}{6}$ (**D**) $\boxed{\frac{x^2+2x-1}{3}}$ (**E**) $\frac{x^2+9x-4}{9}$

28. How many integers between 100 and 1000 are multiples of 7 ?

(**A**) $\boxed{128}$ (**B**) 130 (**C**) 132 (**D**) 134 (**E**) 136

29. The two shortest sides of a right triangle have lengths $\sqrt{3}$ and 2. Let α be the smallest interior angle of this triangle, What is the value of $\sin\alpha$?

(**A**) $\boxed{\sqrt{\frac{3}{7}}}$ (**B**) $\sqrt{\frac{4}{7}}$ (**C**) $\sqrt{\frac{3}{5}}$ (**D**) $\sqrt{\frac{3}{4}}$ (**E**) $\sqrt{\frac{4}{5}}$

30. Which of the following numbers is the largest ?

(**A**) $\sin 15°$ (**B**) $\cos 15°$ (**C**) $\tan 15°$ (**D**) $\frac{1}{\cos 15°}$ (**E**) $\boxed{\frac{1}{\sin 15°}}$

31. If the roots of $x^2-bx+c=0$ are $\sin\frac{\pi}{7}$ and $\cos\frac{\pi}{7}$, then $b^2=$

(**A**) c (**B**) $\boxed{1+2c}$ (**C**) $1+c$ (**D**) $1-c$ (**E**) $1+c^2$

32. How many real numbers are solutions to the equation $x^4 + 4 \mid x \mid = 10$?

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 3 (**E**) 4

33. The following inequalities hold for all positive integers n:

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{4n+1}} < \sqrt{n} - \sqrt{n-1}.$$

What is the greatest integer which is less than $\sum_{n=1}^{24} \frac{1}{\sqrt{4n+1}}$?

(**A**) 2 (**B**) 3 (**C**) $\boxed{4}$ (**D**) 5 (**E**) 6

34. Given that the vertex of the parabola $y = x^2 + 8x + k$ is on the x-axis, what is the value of k?

(**A**) 0 (**B**) 4 (**C**) 8 (**D**) $\boxed{16}$ (**E**) 24

35. There are 30 people in a room. Among them 11 speak French, 24 speak English and 3 speak neither French nor English. How many people in the room speak both French and English?

(**A**) 2 (**B**) 4 (**C**) 6 (**D**) $\boxed{8}$ (**E**) 10

36. Beginning at 5 : 00 pm, how many hours must elapse before the hour hand and the minute hand of a clock are perpendicular to each other?

(**A**) $\frac{1}{5}$ (**B**) $\boxed{\frac{2}{11}}$ (**C**) $\frac{5}{22}$ (**D**) $\frac{4}{23}$ (**E**) $\frac{7}{30}$

37. A drawer contains 64 balls. Each ball is one of 8 colors, and there are 8 balls of each color. If the balls in the drawer are thoroughly mixed and you randomly choose two of them, what is the possibility that these two balls will have the same color?

(**A**) $\frac{1}{7}$ (**B**) $\frac{1}{8}$ (**C**) $\frac{7}{64}$ (**D**) $\frac{9}{64}$ (**E**) $\boxed{\frac{1}{9}}$

38. Let $3^a = 4$, $4^b = 5$, $5^c = 6$, $6^d = 7$, $7^e = 8$ and $8^f = 9$. What is the value of the product $abcdef$?

(**A**) 1 (**B**) $\boxed{2}$ (**C**) $\sqrt{6}$ (**D**) 3 (**E**) $\frac{10}{3}$

39. Which of the following numbers has the smallest value?

(**A**) $\frac{3}{2}$ (**B**) $\boxed{\log_3 2}$ (**C**) $\frac{\pi}{2}$ (**D**) $\log_4 10$ (**E**) $4^{\frac{1}{3}}$

40. If $\sin x = 2\cos x$, what is the value of $\sin 2x$?

(**A**) $\frac{1}{3}$ (**B**) $\frac{2}{5}$ (**C**) $\frac{1}{4}$ (**D**) $\frac{1}{2}$ (**E**) $\boxed{\frac{4}{5}}$

41. A bicyclist riding against the wind averages 10 miles per hour in traveling from A to B, but with the wind averages 15 miles per hour in returning from B to A. How many miles per hour in his average speed for the round trip?

(**A**) 10 (**B**) 11 (**C**) $\frac{25}{2}$ (**D**) $\boxed{12}$ (**E**) 13

42. How many pairs (x, y) of positive integers satisfy $2x + 7y = 1000$?

(**A**) 41 (**B**) 51 (**C**) 61 (**D**) $\boxed{71}$ (**E**) infinity

43. If prices go down by 20%, by what percentage does your purchasing power increase? ("Purchasing power" means the amount of goods that you can purchase for a fixed amount of money.)

(**A**) 20 (**B**) $\boxed{25}$ (**C**) 30 (**D**) 35 (**E**) 40

44. What is the smallest positive integer $n > 150$ such that $\binom{n}{151}$ is divisible by $\binom{n}{150}$ but not equal to it?

(**A**) 201 (**B**) 302 (**C**) 252 (**D**) 352 (**E**) $\boxed{452}$

45. A chemist has 100 *cc* of 20% acid, the rest being water. She adds pure acid to make the solution $33\frac{1}{3}\%$ acid. How many *ccs* of water must she then add to return it to 20% acid?

(**A**) 60 (**B**) 70 (**C**) $\boxed{80}$ (**D**) 90 (**E**) 100

46. If $2^{2^x} + 4^{2^x} = 56$, then what is the value of $2^{2^{2^x}}$?

(**A**) 16 (**B**) 32 (**C**) 64 (**D**) $\boxed{128}$ (**E**) 256

47. If $\log_7 3 = a$ and $\log_7 4 = b$, find x in terms of a and b if $9^x = 28$.

(**A**) $\boxed{x = \frac{1+b}{2a}}$ (**B**) $x = \frac{7b}{a^2}$ (**C**) $x = \frac{b-a}{2}$ (**D**) $x = \frac{2a}{b+1}$ (**E**) none of these

48. 16 college students are going to the beach in 4 identical vans. Each van can hold exactly four students. How many ways can we distribute the students in the vans? (Two distributions are different if there are two students who ride together in one distribution but not in the other. If the same groups of students are together, it does not matter which van they ride in.)

(**A**) 16! (**B**) $(4!)^3$ (**C**) $\binom{16}{4}^3$ (**D**) $\boxed{\binom{16}{4}\binom{12}{4}\binom{8}{4}}$ (**E**) None of the above

49. Let $\otimes$ be an operation defined on functions such that $f \otimes g(x) = f(g(x)) - g(f(x))$. If $f(x) = x^2 - 1$ and $g(x) = 2x + 1$, find $f \otimes g(x)$.

(**A**) $x^2 - 2x - 2$ (**B**) $2x^2 - 4x + 1$ (**C**) $2x^2 + 4x - 2$ (**D**) $\boxed{2x^2 + 4x + 1}$ (**E**) $2x^2 + 2x + 1$

50. Given $-\frac{\pi}{2} < \sin^{-1} x < \frac{\pi}{2}$, then $\tan(\sin^{-1} x)$ must equal to:

(**A**) $\frac{x}{1-x^2}$ (**B**) $\frac{x}{x^2-1}$ (**C**) $\boxed{\frac{x}{\sqrt{1-x^2}}}$ (**D**) $\frac{x}{\sqrt{x^2-1}}$ (**E**) none of these