

2010

ALABAMA

STATEWIDE MATHEMATICS CONTEST



First Round : March 27, 2010  
 Second Round: April 24, 2010 at The University of Alabama

## ALGEBRA II WITH TRIGONOMETRY EXAM

Construction of this test directed  
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### INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D, and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered, and 0 points for each wrong answer. (Thus a “perfect paper” with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- |  |  |
|--|--|
| • If $A$ and $B$ are points, then:                           | • If $A$ is an angle, then:                                  |
| $\overline{AB}$ is the segment between $A$ and $B$           | $m \angle A$ is the measure of angle $A$ in degrees          |
| $\overleftrightarrow{AB}$ is the line containing $A$ and $B$ | • If $A$ and $B$ are points on a circle, then:               |
| $\overrightarrow{AB}$ is the ray from $A$ through $B$        | $\widehat{AB}$ is the arc between $A$ and $B$                |
| $AB$ is the distance between $A$ and $B$                     | $m \widehat{AB}$ is the measure of $\widehat{AB}$ in degrees |

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1. Which of the following is equal to  $\sqrt[4]{128}$ ?
- (A)  $\boxed{2\sqrt[4]{8}}$  (B)  $2\sqrt[4]{9}$  (C)  $4\sqrt[4]{8}$  (D)  $8\sqrt[4]{2}$  (E)  $9\sqrt[4]{2}$
2. If it true that  $a^5 < a < a^4$ , then which of the following must also be true?
- (A)  $-1 < a < 0$  (B)  $0 < a < 1$  (C)  $1 < a < 2$  (D)  $a \geq 2$  (E)  $\boxed{a < -1}$
3. How many polynomials are there of the form  $x^3 - 8x^2 + cx + d$  such that  $c$  and  $d$  are real numbers and the three roots of the polynomial are distinct positive integers?
- (A) 1 (B)  $\boxed{2}$  (C) 3 (D) 4 (E) 5
4. Let  $f(x)$  be a function such that, for every real number  $x$ ,  $f(-x) + 2f(x) = \cos x$ . What is the value of  $f(\frac{\pi}{4})$ ?
- (A)  $\sqrt{2}$  (B)  $\frac{1}{2}$  (C)  $\frac{\sqrt{3}}{2}$  (D)  $\frac{\sqrt{3}}{4}$  (E)  $\boxed{\frac{\sqrt{2}}{6}}$
5. Let  $g(x) = \frac{a^x + a^{-x}}{a^x - a^{-x}}$ , where  $a > 0$  and  $a \neq 1$ . If  $g(p) = 2$ , then  $g(2p) = ?$
- (A) 2 (B) 3 (C)  $\sqrt{2}$  (D)  $\boxed{\frac{5}{4}}$  (E)  $\frac{8}{7}$
6. If  $i^2 = -1$ , then what is the value of  $i^{2010}$ ?
- (A) 1 (B)  $i$  (C)  $\boxed{-1}$  (D)  $-i$  (E) None of these
7. Determine the coefficient of  $x^6$  in the expansion of  $\left(\frac{x^2}{4} + \frac{2}{x}\right)^{12}$ . The collect answer is  $\frac{231}{16}$ .
- (A) 80 (B) 81 (C) 82 (D) 83 (E) 84
8. The graphs of the two lines  $\begin{matrix} x + 3y = 0 \\ ax + y = 0 \end{matrix}$  are perpendicular to each other. What is the value for  $|a|$ ?
- (A)  $\frac{1}{3}$  (B)  $-\frac{1}{3}$  (C) 1 (D)  $\boxed{3}$  (E) -3
9. Let A be the positive integer satisfying the following equation  $\frac{9}{10} + \frac{99}{10^2} + \frac{999}{10^3} + \dots + \frac{9 \dots 9}{10^{10}} = \frac{A}{10^{10}}$ . How many 8's appear in the decimal representation of A?
- (A) 0 (B) 1 (C) 7 (D)  $\boxed{9}$  (E) 11
10. Which of the following is equal to  $\sin^{\frac{1}{2}} x \cos x - \sin^{\frac{5}{2}} x \cos x$ ?
- (A)  $\sec^5 x \tan^3 x$  (B)  $\boxed{\cos^3 x \sqrt{\sin x}}$  (C)  $\cos^2 x \sin^{\frac{1}{2}} x$  (D)  $\cos^3 x \sin^2 x$  (E) None of these

11. The limit of  $\frac{x^2 - 1}{x - 1}$  as  $x$  approaches 1 is  
 (A) 0 (B) indeterminate (C)  $x - 1$  (D)  $\boxed{2}$  (E) 1
12. What is the slope of the line that goes through points  $(-3, 2)$  and  $(3, 2)$ ?  
 (A) Undefined (B)  $\boxed{0}$  (C)  $\frac{2}{3}$  (D)  $-\frac{3}{2}$  (E) 1
13. For which value of  $\theta$  listed below is it true that  $2^{\cos \theta} > 1$  and  $3^{\sin \theta} < 1$ ?  
 (A)  $10^\circ$  (B)  $80^\circ$  (C)  $150^\circ$  (D)  $220^\circ$  (E)  $\boxed{290^\circ}$
14. Let  $A$  denote the area of the region described by the inequality  $|x|^3 + |y|^3 < 1$ . Then  $A$  is in which of the following ranges?  
 (A)  $1 < A < \frac{\pi}{2}$  (B)  $\frac{\pi}{2} < A < 2$  (C)  $2 < A < \pi$  (D)  $\boxed{\pi < A < 4}$  (E)  $4 < A < 2\pi$
15. Suppose  $x$  is a complex number for which  $x + \frac{1}{x} = 2 \cos 12^\circ$ . What is the value of  $x^5 + \frac{1}{x^5}$ ?  
 (A)  $\boxed{1}$  (B) 2 (C) 3 (D) 4 (E) 5
16. What's the number of sides of a regular polygon for which the number of diagonals is 43 or 44?  
 (A) 8 (B) 9 (C) 10 (D)  $\boxed{11}$  (E) 12
17. Let  $Z = \frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \frac{1}{f}}}}}}$ . If  $a, b, c, d, e$  and  $f$  are all either 1 or 2. What is the maximum value of  $Z$ ? The correct answer is  $\frac{30}{41}$ .  
 (A) 1 (B) 3 (C) 5 (D) 7 (E) 9
18. Given the function  $f(x) = 5x^3 - 7x^2 + 11x + 13$ , what is the sum of  $A, B, C$  and  $D$  where  $f(x) = A(x + 1)^3 + B(x + 1)^2 + C(x + 1) + D$ ?  
 (A) 10 (B) 11 (C) 12 (D)  $\boxed{13}$  (E) 14
19. Suppose  $a, b > 1$ . If  $a^x = b^q$  and  $b^y = a^z$ , then  
 (A)  $\boxed{xy = qz}$  (B)  $\frac{x}{y} = \frac{q}{z}$  (C)  $x + y = q + z$  (D)  $x - y = q - z$  (E)  $x^y = q^z$
20. What are the factors of a polynomial function if its graph has  $x$ -intercepts at  $-1, 0$ , and  $4$ ?  
 (A)  $x, (x - 1), (x + 4)$  (B)  $\boxed{x, (x + 1), (x - 4)}$   
 (C)  $(x - 1), (x - 4)$  (D)  $(x - 1), (x + 4)$  (E)  $x, (x + 4)$

21. If  $S = i^n + i^{-n}$  (where  $i^2 = -1$ ) and  $n$  is an integer, then the total number of possible distinct values for  $S$  is  
 (A) 1 (B) 2 (C) 3 (D) 4 (E)  $\infty$
22. Evaluate  $\cos(\frac{\pi}{2} - x) \csc x$ .  
 (A) 0 (B) 2 (C) 3 (D)  $\frac{\pi}{2}$  (E) 1
23. Given the sequence  $2^5 - 2, 3^5 - 3, 4^5 - 4, \dots, n^5 - n, \dots$ . The largest of the following integers which divides all the members of the sequence is  
 (A) 1 (B) 2 (C) 6 (D) 15 (E) 30
24. Define a sequence by  $b_1 = 2$  and  $b_{n+1} = \frac{1 - b_n}{1 + b_n}$ , for  $n \geq 1$ , what is the value of  $b_{2010}$ ?  
 (A) -2010 (B) 2010 (C) 2000 (D) -1 (E)  $-\frac{1}{3}$
25. Which expression correctly represents "\$10 less than twice the casts  $C$ "?  
 (A)  $10 - 2C$  (B)  $8 + C$  (C)  $2C - 10$  (D)  $\frac{C}{2} - 10$  (E)  $10 + 2C$
26. If  $10^{2y} = 25$ , then  $10^{-y}$  equals  
 (A)  $-\frac{1}{5}$  (B)  $\frac{1}{625}$  (C)  $\frac{1}{50}$  (D)  $\frac{1}{25}$  (E)  $\frac{1}{5}$
27. Evaluate  $\sin 67^\circ \sin 22^\circ + \sin 23^\circ \sin 68^\circ$ .  
 (A)  $\frac{1}{\sqrt{2}}$  (B)  $\sin 89^\circ$  (C) 0 (D)  $\sqrt{2}$  (E)  $\frac{1}{2}$
28. A line with slope 3 intersects a line with slope 6 at the point (30, 40). What is the distance between the y-intercepts of these lines?  
 (A) 80 (B) 90 (C) 100 (D) 110 (E) 120
29. Consider the equation:  $x^2 + px + q = 0$ . If the roots of this equation differ by 2, then  $q$  equals  
 (A)  $p$  (B)  $\frac{(p+1)^2}{4}$  (C)  $\frac{p^2+4}{4}$  (D)  $\frac{p^2-4}{4}$  (E)  $\frac{(p-1)^2}{4}$
30. The matrix equation which represents the linear system  $3x - 2y = 5, 8x + 9y = -2$  is  
 (A)  $\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 8 & 9 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$  (B)  $\begin{pmatrix} 3 & 8 \\ -2 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$  (C)  $\begin{pmatrix} 3 & -2 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} xy \\ xy \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$   
 (D)  $\begin{pmatrix} 3 & 9 \\ -2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$  (E)  $\begin{pmatrix} 3 & -2 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

31. If the equation  $x^2 - bx + c = 0$  has two real solutions,  $x = p$  and  $x = q$ , and  $p \geq q$ , then  $p - q = ?$   
 (A)  $b^2 - 4c$       (B)  $b^2 - 2c$       (C)  $\sqrt{b^2 - 4c}$       (D)  $\sqrt{b^2 + 4c}$       (E)  $\sqrt{b^2 - 2c}$
32. Find the largest positive number  $\delta$  such that  $|\sqrt{x} - 2| < 0.1$  whenever  $|x - 4| < \delta$ .  
 (A)  $\delta = 0.2$       (B)  $\delta = 0.39$       (C)  $\delta = 0.4$       (D)  $\delta = 0.41$       (E)  $\delta = 0.6$
33. Which of the following is equivalent to  $\frac{x^5 y^2 z^8}{(xy)^{-3}}$ ?  
 (A)  $\frac{x^2 z^8}{y}$       (B)  $x^{12} y^8 z^8$       (C)  $\frac{x^{-4} y z^8}{3}$       (D)  $x^8 y^5 z^8$       (E)  $x^{-2} y z^8$
34. If  $\frac{1}{a-x} = \frac{1}{a} - \frac{1}{x}$  and  $a$  is a non-zero constant. How many real number solutions will this equation have?  
 (A) 0      (B) 1      (C) 2      (D) 3      (E) 4
35. Given  $y = x^2 - x - 1$ ,  $y = -x^2 + 4x - 1$ , which set of the following ordered pairs is the solution to the system?  
 (A)  $\{(0, 3), (-1, 2)\}$       (B)  $\{(0, -1), (3, 2)\}$       (C)  $\{(0, -1), (6, 23)\}$   
 (D)  $\{(3, 2), (6, -11)\}$       (E) Non of these
36. Write  $x, y$  and  $z$  in order from largest to smallest if  $x = 2^{100}$ ,  $y = 3^{75}$  and  $z = 5^{50}$ .  
 (A)  $x > y > z$       (B)  $x > z > y$       (C)  $y > x > z$       (D)  $y > z > x$       (E)  $z > x > y$
37. An acute angle is formed by two lines of slope 1 and 3. What is the slope of the line which bisects this angle?  
 (A)  $1/2$       (B) 2      (C)  $\frac{(1 + \sqrt{5})}{2}$       (D)  $\sqrt{5}$       (E)  $\sqrt{7}$
38. Suppose that  $f(n+1) = f(n) + f(n-1)$  for  $n = 1, 2, \dots$ . Given that  $f(6) = 2$  and  $f(4) = 8$ , what is  $f(3) + f(5)$ ?  
 (A) -18      (B) -19      (C) -20      (D) -21      (E) -22
39. The price of an item is reduced by 20%. By what percentage must the price increase to return to the original one?  
 (A) 20%      (B) 22.5%      (C) 25%      (D) 27.5%      (E) 30%
40. Solve for  $x$ :  $8^x + 7^x = y$ .  
 (A)  $x = 2$       (B)  $x = \frac{\ln y}{7 \ln 7}$       (C)  $x = \frac{\ln y - \ln 7}{\ln 8}$       (D)  $x = \frac{\ln y + 7}{\ln 8}$       (E) None of these

41. Determine the equation of the line passing through the points of intersection of the circles  $(x + 2)^2 + (y + 1)^2 = 9$  and  $(x + 3)^2 + (y + 4)^2 = 16$ .
- (A)  $2x + 6y = -13$  (B)  $10x - 6y = -7$   
(C)  $-2x + 6y = -7$  (D)  $-10x + 6y = 25$  (E)  $2x - 6y = 13$
42. If the sum of the first  $n$  terms of a sequence is  $n^2 + 5n$ , what is the 2010<sup>th</sup> term of the sequence?
- (A) 4042 (B)  $4024$  (C) 2042 (D) 4240 (E) 2240
43. How many solutions do the trigonometric equation  $\frac{\sin x}{1 + \cos x} = 1$  has in the interval  $[\pi, 2\pi]$ ? The correct answer is 0.
- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5
44. How many pairs of positive integers  $(n, m)$  with  $n \geq m$  satisfy the following equation:  $\frac{1}{5} = \frac{1}{n} + \frac{1}{m}$ ?
- (A) 0 (B) 1 (C)  $2$  (D) 3 (E) 4
45. Let  $P(x) = x^3 - 4x^2 + 8x - 11$  have roots  $a, b,$  and  $c$ . What is the value of the expression  $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac}$ ?
- (A)  $\frac{4}{11}$  (B)  $\frac{8}{11}$  (C)  $-\frac{4}{11}$  (D)  $-\frac{8}{11}$  (E)  $\frac{12}{11}$
46. If  $z$  is a complex number satisfying  $z^2 + z + 1 = 0$ , then  $z^{12} = ?$
- (A)  $1$  (B)  $-1$  (C)  $i$  (D)  $16$  (E)  $16i$
47. If  $x$  is positive and  $\log x \geq \log 2 + \frac{1}{2} \log x$ , then which of the following statement is correct?
- (A)  $x$  has no minimum or maximum value (B) the maximum value of  $x$  is 1  
(C) the minimum value of  $x$  is 1 (D) the maximum value of  $x$  is 4 (E)  $\frac{1}{2}$
48. If  $4^x - 4^{x-1} = 24$ , then the value of  $(2x)^x$  equals
- (A)  $5\sqrt{5}$  (B)  $\sqrt{5}$  (C)  $25\sqrt{5}$  (D) 125 (E) 25
49. Suppose  $m$  and  $n$  are positive integers with  $m < n$ . The value of  $\frac{m+5}{n+5}$  will be
- (A) 1 (B) 3 (C)  $< \frac{m}{n}$  (D)  $> \frac{m}{n}$  (E)  $\frac{m}{n}$
50. Express the rational number  $s = \sin^2(10^\circ) + \sin^2(20^\circ) + \dots + \sin^2(80^\circ)$  in the lowest terms.
- (A) 1 (B) 2 (C) 3 (D)  $\frac{4}{5}$  (E) 5