2005

ALABAMA

STATEWIDE MATHEMATICS CONTEST

DIVISION I COMPREHENSIVE EXAM

Construction of this text directed
by
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1. Suppose a and b are integers. Find $a+b$ if $\sqrt{11-6\sqrt{2}} = a + b\sqrt{2}$.

(**A**) 4 (**B**) 3 (**C**) $\boxed{2}$ (**D**) 1 (**E**) 0

2. Which one of the following is **not** a factor of $x^6 - 1$?

(**A**) $x-1$ (**B**) x^2-1 (**C**) x^2+x+1 (**D**) x^4+x^2+1 (**E**) $\boxed{\text{All are factors}}$

3. How many positive integers less than 70 are relatively prime to 70 ?

(**A**) 22 (**B**) $\boxed{24}$ (**C**) 26 (**D**) 28 (**E**) 30

4. What is the remainder if 2^{2005} is divided by 13 ?

(**A**) 0 (**B**) 1 (**C**) $\boxed{2}$ (**D**) 4 (**E**) 7

5. How many consecutive zeros does $\dfrac{100!}{2^{19}5^{10}}$ have at the end?

(**A**) 8 (**B**) 10 (**C**) 12 (**D**) $\boxed{14}$ (**E**) 16

6. If $x + \dfrac{1}{x} = 3$ and $x > 1$, what is the value of $x^3 - \dfrac{1}{x^3}$?

(**A**) $\boxed{8\sqrt{5}}$ (**B**) $-5\sqrt{2}$ (**C**) $-8\sqrt{5}$ (**D**) $5\sqrt{2}$ (**E**) None of these

7. If $23^a = 5$, $23^b = 10$, what is the value of $2^{\frac{1}{b-a}}$?

(**A**) 5 (**B**) 10 (**C**) 15 (**D**) 20 (**E**) $\boxed{23}$

8. Let $f(x) = \log(x + \sqrt{x^2+1})$. Which one of the following statements are true?
I. The domain of $f(x)$ is all real numbers.
II. The graph of $f(x)$ contains the origin $(0,0)$.
III. $f(-x) = -f(x)$.

(**A**) I (**B**) II (**C**) I and II (**D**) II and III (**E**) $\boxed{\text{I, II and III}}$

9. If $p = \dfrac{1}{\sqrt{14}-\sqrt{13}}$ and $q = \dfrac{1}{\sqrt{14}+\sqrt{13}}$, what is $p^2 + pq + q^2$?

(**A**) 49 (**B**) 52 (**C**) $\boxed{55}$ (**D**) 58 (**E**) 61

10. What is the sum of the coefficients of all the terms of $(1+2x+3x^2-4x^3)^{10}$ when it is expanded?

(**A**) 4 (**B**) 16 (**C**) 256 (**D**) $\boxed{1024}$ (**E**) 2048

11. If $f(\sqrt{t-3}) = \dfrac{t+1}{t-1}$, what is $f(2)$?

(**A**) 3 (**B**) $\boxed{\dfrac{4}{3}}$ (**C**) $\dfrac{5}{4}$ (**D**) $\dfrac{6}{5}$ (**E**) -3

12. Find sum of the minimum distance and the maximum distance from the point $(4, -3)$ to the circle $x^2 + y^2 + 4x - 10y - 7 = 0$.

(**A**) 10 (**B**) 15 (**C**) $\boxed{20}$ (**D**) 25 (**E**) 30

13. If both of the functions $f(x) = \sin x$ and $g(x) = \cos x$ are defined on $[0, \frac{\pi}{2}]$, what is $f\left(f^{-1}\left(\frac{1}{3}\right) + g^{-1}\left(\frac{1}{3}\right)\right)$?

(**A**) $\boxed{1}$ (**B**) $\frac{1}{3}$ (**C**) $\frac{1}{9}$ (**D**) $\frac{1}{2}$ (**E**) None of these

14. If $i = \sqrt{-1}$, what is the value of $\left(\frac{1+i}{1-i}\right)^{2005}$?

(**A**) 0 (**B**) 1 (**C**) -1 (**D**) $\boxed{i}$ (**E**) $-i$

15. If the equality $x^3 = A(x-1)(x-2)(x-3) + B(x-1)(x-2) + C(x-1) + D$ is an identity, what is $A + B + C + D$?

(**A**) 13 (**B**) $\boxed{15}$ (**C**) 17 (**D**) 19 (**E**) 21

16. If the height of a cylinder is increased by 10%, and the radius of the cylinder is decreased by 10%, what will happen to the volume of the cylinder?

(**A**) Same as the volume of the original cylinder (**B**) Increases by 8.9%

(**C**) Decreases by 8.9% (**D**) Increases by 10.9% (**E**) $\boxed{\text{Decreases by 10.9\%}}$

17. If $f(x) = \dfrac{x^2}{x^2 - 1}$, what is $51 \cdot f(50) \cdot f(49) \cdot \cdots \cdot f(3) \cdot f(2)$?

(**A**) $\boxed{100}$ (**B**) 120 (**C**) 140 (**D**) 160 (**E**) 180

18. Let x denote the following infinite series: $x = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$ Express the series $y = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots$ in terms of x.

(**A**) $2x$ (**B**) $\boxed{\frac{4}{3}x}$ (**C**) $\frac{5}{3}x$ (**D**) $\frac{6}{5}x$ (**E**) $\frac{7}{5}x$

19. Find $B + 2C$ if the quadratic function $f(x) = Ax^2 + Bx + C$ has the minimum value 11 when $x = 1$.

(**A**) $\boxed{22}$ (**B**) 20 (**C**) 18 (**D**) 16 (**E**) 14

20. Let $O(0, 0)$, $A(4, 13)$ and $B(8, 9)$ be three points on the xy-plane. If the line $y = kx$ bisects the area of the triangle OAB, what is k ?

(**A**) $\frac{3}{2}$ (**B**) $\frac{5}{3}$ (**C**) $\frac{9}{5}$ (**D**) $\boxed{\frac{11}{6}}$ (**E**) None of these

21. Find the remainder when the sum $\sum_{n=1}^{10} \left(n^2+3n+1\right)\cdot n!$ is divided by 10.

(**A**) 1 (**B**) 3 (**C**) 5 (**D**) $\boxed{7}$ (**E**) 9

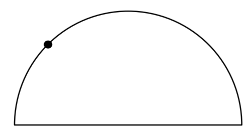
22. Suppose that a function $f(x)$ defined on natural numbers satisfies following conditions:
(i) $f(1)=1$
(ii) $f(x+y)=f(x)+f(y)+xy$ for all x, y
What is the value of $f(1)+f(2)+f(3)+f(4)$?

(**A**) 10 (**B**) 15 (**C**) $\boxed{20}$ (**D**) 25 (**E**) 30

23. If $\sin^{16} a = \frac{1}{5}$, what is $\frac{1}{\cos^2 a}+\frac{1}{1+\sin^2 a}+\frac{2}{1+\sin^4 a}+\frac{4}{1+\sin^8 a}$?

(**A**) 4 (**B**) 6 (**C**) 8 (**D**) $\boxed{10}$ (**E**) None of these

24. There are seven points plotted on a semicircle as in the figure. How many triangles can be made by joining the points in the semicircle?



(**A**) 30 (**B**) $\boxed{31}$ (**C**) 32 (**D**) 33 (**E**) None of these

25. How many solutions are there for the equation $\cos^2(x)-\sin^2(2x)=0$ on $[0,2\pi]$?

(**A**) $\boxed{6}$ (**B**) 4 (**C**) 2 (**D**) 1 (**E**) None of these

26. Which ones of the following statements are true?
(i) $\log_2(n+3) > \log_2(n+2)$. (ii) $\log_2(n+2) > \log_3(n+2)$. (iii) $\log_2(n+2) > \log_3(n+3)$.

(**A**) (i) (**B**) (ii) (**C**) (i) and (ii) (**D**) (i) and (iii) (**E**) $\boxed{\text{(i),(ii) and (iii)}}$

27. Imagine that you have two thumbtacks placed at two points, A and B. If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse. What is the best way to maximize the area surrounded by the ellipse with a fixed length of string?
I Take the two points A and B apart so that they have the maximum distance.
II Make two points A and B coincide.
III Place A and B vertically.
IV Place A and B horizontally.
V The area is always same regardless of the location of A and B.

(**A**) I (**B**) $\boxed{\text{II}}$ (**C**) III (**D**) IV (**E**) V

28. Find the sum of the following series: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{9}+\frac{1}{8}+\frac{1}{27}+\frac{1}{16}+\cdots$

(**A**) $\frac{7}{3}$ (**B**) $\boxed{\frac{5}{2}}$ (**C**) $\frac{7}{4}$ (**D**) $\frac{5}{3}$ (**E**) None of these

29. Suppose that 10 straight lines are drawn on a piece of paper so that every pair of lines intersects, but no three lines intersect at a common point. Into how many regions do the 10 straight lines divide the plane ?

(**A**) 50 (**B**) 51 (**C**) 53 (**D**) 55 (**E**) $\boxed{56}$

30. Let α and β be the solutions to the equation $2x^2 - x - 2 = 0$. Find the value of $2\alpha + 2\beta + (\alpha\beta)^{2005}$?

(**A**) $\boxed{0}$ (**B**) 1 (**C**) 2 (**D**) -1 (**E**) -2

31. Michelle decides to start saving money. She plans to save 1 cent the first day of January, 4 cents the second day, 10 cents the third day, 19 cents the fourth day, 31 cents the fifth day, 46 cents the sixth day, and so on for the month. How much will she save in January? There are 31 days in January.

(**A**) \$13.96 (**B**) \$14.71 (**C**) \$135.16 (**D**) $\boxed{\$149.11}$ (**E**) None of these

32. Nicole is making a bracelet putting beads on a string in order. If she has 10 different beads, how many different bracelets are possible ?

(**A**) $\boxed{\dfrac{9!}{2}}$ (**B**) $\dfrac{10!}{2}$ (**C**) $9!$ (**D**) $10!$ (**E**) $11!$

33. Among the following shapes of equal perimeter, which one has the largest area ?

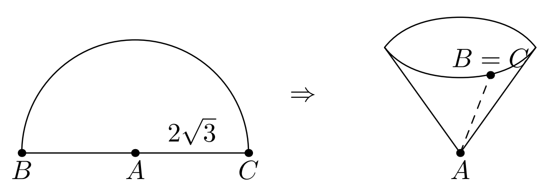
(**A**) $\boxed{\text{circle}}$ (**B**) equilateral (**C**) square (**D**) ellipse (**E**) regular pentagon

34. How many three digit numbers are same when the first and the last digit are interchanged ?

(**A**) 1000 (**B**) 729 (**C**) 100 (**D**) $\boxed{90}$ (**E**) 81

35. Eric wants to construct a conical paper cup by gluing together two sides $\overline{AB}$ and $\overline{AC}$ in a semi circle with radius, $AB = AC = 2\sqrt{3}$. What is the volume of the cup ?

(**A**) π (**B**) $2\sqrt{3}\pi$ (**C**) $4\sqrt{3}\pi$ (**D**) $\boxed{3\pi}$ (**E**) 9π



36. Let x be $1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}}$. Evaluate $(2x-1)^2$.

(**A**) 1 (**B**) 5 (**C**) -2 (**D**) 7 (**E**) $\boxed{3}$

37. Define the sequence of numbers as follows: $f_1 = \log_2 \sqrt{2}$, $f_2 = \log_3 \sqrt{3\sqrt{3}}$, $f_3 = \log_4 \sqrt{4\sqrt{4\sqrt{4}}}$, $f_4 = \log_5 \sqrt{5\sqrt{5\sqrt{5\sqrt{5}}}}, \cdots$. Find the n-th term f_n.

(**A**) 1 (**B**) $\log_{n+1} \sqrt{n+2}$ (**C**) $\boxed{\dfrac{2^n - 1}{2^n}}$ (**D**) $\dfrac{1}{\log_n(n+2)}$ (**E**) $2^n\pi$

38. Let $X = \{1, 2, 3, 4\}$. How many functions $f : X \to X$ are there satisfying $(f \circ f)(x) = x$ for all x in X ?

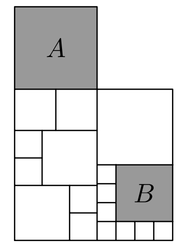
(**A**) 10 (**B**) 11 (**C**) 12 (**D**) $\boxed{13}$ (**E**) None of these

39. Find the area of the region consisting of all points (x, y) so that $x^2 + y^2 \leq 1 \leq |x| + |y|$.

(**A**) π (**B**) $\pi - 1$ (**C**) $\boxed{\pi - 2}$ (**D**) $\pi - 3$ (**E**) $\pi + 1$

40. The following figure is obtained joining squares. Find the ratio of the side of the square A to the one of the square B.

(**A**) 4:3 (**B**) 8:5 (**C**) 15:12 (**D**) $\boxed{16:11}$ (**E**) 17:3



41. Holly wants to make a plan for January next year. She has a calendar for this year, but doesn't have one for next year. Which month of this year shows the same date and day as January of next year?

(**A**) January (**B**) March (**C**) $\boxed{\text{May}}$ (**D**) August (**E**) October

42. Brian paid \$50 for 12 of honeydews, coconut and watermelons. The unit price for honeydew, coconut and watermelon is \$2, \$5 and \$9, respectively. Suppose that Brian bought at least one fruit for each. How many coconuts did Brian buy?

(**A**) 3 (**B**) $\boxed{4}$ (**C**) 5 (**D**) 6 (**E**) 7

43. Suppose that two lines, $y = ax$ and $y = bx$ are symmetric with respect to the line $y = x$. If the angle between two lines is $20°$, what is the value of $3ab$?

(**A**) 1 (**B**) 2 (**C**) $\boxed{3}$ (**D**) 4 (**E**) 5

44. When the polynomial $f(x)$ is divided by $x + 3$, the quotient is $x^2 - 1$ and the remainder is 2. Find the remainder when this polynomial $f(x)$ is divided by $x - 2$.

(**A**) 1 (**B**) 2 (**C**) 15 (**D**) $\boxed{17}$ (**E**) 20

45. Let a and b be the coefficients of x^3 in $(1+x+2x^2+3x^3)^3$ and $(1+x+2x^2+3x^3+4x^4)^3$, respectively. Find the value of $a - b$.

(**A**) $\boxed{0}$ (**B**) 1 (**C**) -1 (**D**) $3^3 - 4^3$ (**E**) $3^3 - 3^4$

46. The line $(k + 1)^2 x + ky - 2k^2 - 2 = 0$ passes through a point regardless of the value k. Which of the following is the line with slope 2 passing through the point?

(**A**) $\boxed{y = 2x - 8}$ (**B**) $y = 2x - 5$ (**C**) $y = 2x - 4$ (**D**) $y = 2x + 5$ (**E**) $y = 2x + 8$

47. What percentage of the interval $[-10, 10]$ is inequality $x + 2 > \frac{5}{x-2}$ satisfied?

(**A**) 70% (**B**) $\boxed{60\%}$ (**C**) 50% (**D**) 40% (**E**) 30%

48. What is the radius of a circle which circumscribes the triangle with vertices $(2,0)$, $(0,4)$, and $(4,6)$?

(**A**) $\sqrt{5}$ (**B**) $\boxed{\sqrt{10}}$ (**C**) 1 (**D**) $2\sqrt{5}$ (**E**) $\sqrt{40}$

49. Solve $3^x + 9^x = y$ for x.

(**A**) $x = 3$ (**B**) $x = \dfrac{\ln y}{2\ln 3}$ (**C**) $x = \dfrac{\ln y - \ln 3}{\ln 9}$ (**D**) $x = \dfrac{\ln y + 3}{\ln 9}$ (**E**) $\boxed{\text{None of these}}$

50. John tries to prove his conjecture that $\dfrac{f(a)}{a-b}$ is integer for two distinct integers, a and b for the given polynomial $f(x)$ which satisfies the following two properties, (I) and (II).
(I) Every coefficient in $f(x)$ is an integer. (II) $f(a)f(b) = -(a-b)^2$.
It is well known that
(III) For given integers m and n, if the solutions of a quadratic equation $x^2 + mx + n = 0$ are rational numbers, then they are integers.
Find out a mistake from his proof if there is any.
Proof:
(a) For a natural number n, $f(a) - f(b)$ is divisible by $a - b$ since $a^n - b^n$ is divisible by $a - b$ by (I).
(b) Thus $\dfrac{f(a)-f(b)}{a-b}$ is integer. The quadratic equation which has two solutions $\dfrac{f(a)}{a-b}$ and $\dfrac{-f(b)}{a-b}$ is $x^2 - \dfrac{f(a)-f(b)}{a-b}x + 1 = 0$ by (II).
(c) Since $\dfrac{f(a)}{a-b}$ is rational by (I), and
(d) $\dfrac{f(a)-f(b)}{a-b}$ is integer, $\dfrac{f(a)}{a-b}$ is integer by (III).

(**A**) (a) (**B**) (b) (**C**) (c) (**D**) (d) (**E**) $\boxed{\text{None}}$