

20  
07

ALABAMA

STATEWIDE MATHEMATICS CONTEST



First Round : March 31, 2007  
 Second Round: April 21, 2007 at The University of Alabama

## DIVISION I COMPREHENSIVE EXAM

Construction of this test directed  
 by  
 Paul Allen, The University of Alabama

### INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D, and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered, and 0 points for each wrong answer. (Thus a “perfect paper” with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one or more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- |   |   |
|---|---|
| <ul style="list-style-type: none"> <li>• If <math>A</math> and <math>B</math> are points, then:<br/> <math>\overline{AB}</math> is the segment between <math>A</math> and <math>B</math><br/> <math>\overleftrightarrow{AB}</math> is the line containing <math>A</math> and <math>B</math><br/> <math>\overrightarrow{AB}</math> is the ray from <math>A</math> through <math>B</math><br/> <math>AB</math> is the distance between <math>A</math> and <math>B</math></li> </ul> | <ul style="list-style-type: none"> <li>• If <math>A</math> is an angle, then:<br/> <math>m \angle A</math> is the measure of angle <math>A</math> in degrees</li> <li>• If <math>A</math> and <math>B</math> are points on a circle, then:<br/> <math>\widehat{AB}</math> is the arc between <math>A</math> and <math>B</math><br/> <math>m \widehat{AB}</math> is the measure of <math>\widehat{AB}</math> in degrees</li> </ul> |
|---|---|

Editing by Zhijian Wu, The University of Alabama  
 Printing by The University of Alabama

# What You Can Do With A Mathematics Major

## Occupational opportunities

Actuarial and Insurance	Government	Accountant
Computer & Information Sciences	Investment Analyst	Financial Planner
Researcher	Benefits Specialist	Mathematician
Demographers	Computer Programmer	Cartographer
Data Processor	Navigator	Meteorologist
Applications Programmer	Ecologist	Health
Systems Analyst	Biomedical Engineer	Bio-mathematician
Computer Applications Engineer	Operations Analyst	Operations Research
Control Systems Engineer	Control Systems Engineer	Systems Engineer
Numerical Analyst	Teaching	Business Industry
Statistician	Engineering Analyst	Financial Analyst
Technical Writer	Homeland Security	Communications Engineer

Study in the field of mathematics offers an education with an emphasis on careful problem analysis, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, health and environmental fields, and the academic world require sophisticated mathematical techniques for their solution. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of Alabama offers undergraduate and graduate degrees in Mathematics. Please visit [www.ua.edu](http://www.ua.edu) and refer to the undergraduate and graduate programs for additional information.

## Engineering Math Advancement Program

The University of Alabama is offering a summer program to build math skills for students entering engineering. The Engineering Math Advancement Program, E-MAP, is a five-week summer residence class that addresses math and engineering prerequisites for incoming engineering students. The program targets bright students who may not have retained the information learned in high school and provides an opportunity to hone technical abilities before entering college. The goal of E-MAP is to assist entering freshmen in developing a solid background in calculus to succeed in engineering before they start at the University.

Classes are designed around Precalculus Algebra and Trigonometry and incorporate important learning principles to ensure that knowledge is retained and not just memorized. Students develop their skills through hands-on experiences, problem solving teaming exercises, and interaction with engineering professors and instructors through an interdisciplinary Living Laboratory program. Experiments allow students to use simple calculus in engineering applications. The program also involves introducing students to local practicing engineers through work on one or more community service engineering-related activities. E-MAP will reserve 33-40 percent of enrollment space for underrepresented groups. Financial assistance is available based on need. Please visit [www.emap.ua.edu](http://www.emap.ua.edu) for additional information.

1. Find the sum of the following series:

$$\frac{1}{3} - \frac{1}{2} + \frac{1}{9} - \frac{1}{4} + \frac{1}{27} - \frac{1}{8} + \dots$$

- (A)  $-\frac{1}{3}$       (B) 0      (C)  $\boxed{-\frac{1}{2}}$       (D)  $\frac{1}{5}$       (E)  $-\frac{2}{3}$

2. How many *distinct* roots does  $p(x) = x^7 + 3x^6 + 5x^5 + 7x^4 + 7x^3 + 5x^2 + 3x + 1$  have?

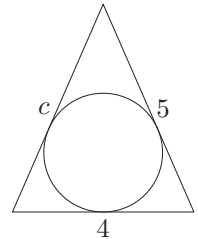
- (A)  $\boxed{3}$       (B) 4      (C) 5      (D) 6      (E) 7

3. The equation  $x^3 = 46 + 9i$  where  $i = \sqrt{-1}$  has a solution of the form  $a + bi$  where  $a$  and  $b$  are integers. Find  $a + b$ .

- (A) 5      (B) -1      (C) 3      (D)  $\boxed{1}$       (E) none of these

4. There are *two possible* triangles with sides 4 and 5 having an inscribed circle of radius 1 (see the figure shown at the right). Which of the following is one of the possible values for the length  $c$ ?

- (A)  $\boxed{3 + 2\sqrt{6}}$       (B) 6      (C)  $1 + \sqrt{20}$       (D) 7      (E)  $2 + \sqrt{5}$



5. Which one of the following is most likely the number of digits in the number  $64^{64}$ ?

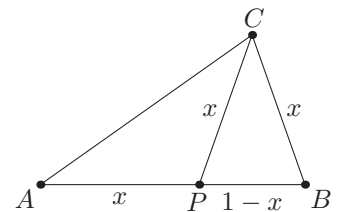
- (A) 78      (B) 84      (C) 192      (D) 144      (E)  $\boxed{116}$

6. Which *two* of the following cannot be factored using integer coefficients?

I.  $x^4 + x^2 + 1$       II.  $x^4 + 2x + 2$       III.  $x^4 - 2x^2 + 1$       IV.  $x^4 + x + 1$

- (A) I and II      (B) I and IV      (C) II and III      (D)  $\boxed{\text{II and IV}}$       (E) III and IV

7. Line segment  $\overline{AB}$  of length 1 has been divided at point  $P$  into a longer segment of length  $x$  and a shorter segment of length  $1 - x$  where  $\frac{1}{x} = \frac{x}{1-x}$ . Moreover, an isosceles triangle  $PCB$  has been constructed with equal legs of length  $x$  as shown in the figure. What is the length of side  $\overline{AC}$ ?



- (A)  $\frac{\sqrt{5} + 1}{2}$       (B)  $\frac{2}{\sqrt{5}}$       (C)  $\sqrt{5} - 1$   
 (D)  $\frac{\sqrt{5} - 1}{2}$       (E)  $\boxed{\text{none of these}}$

8. If 100 bushels of corn is distributed among 100 people in such a manner that each man receives 3 bushels, each woman 2, and each child  $\frac{1}{2}$  of a bushel, which one of the following is a *possible value* for the number of men?

- (A) 16      (B) 12      (C)  $\boxed{8}$       (D) 4      (E) 0

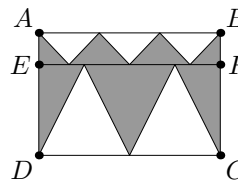
9. The number  $N = 173889$  is a perfect square. What is the sum of the digits in  $\sqrt{N}$ ?  
 (A) 12                      (B) 7                      (C) 14                      (D) 9                      (E) none of these

10. Let  $x = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}}$  be the indicated continued fraction. Which one of the following is equal to  $x$ ?

- (A)  $\frac{\sqrt{2}+1}{4}$                       (B)  $\frac{\sqrt{5}-1}{2}$                       (C)  $\frac{1}{3}$                       (D)  $\frac{\sqrt{6}+1}{4}$                       (E)  $\frac{\sqrt{3}-1}{2}$

11. In the figure shown,  $ABCD$  and  $CDEF$  are rectangles,  $AB = 4$  and  $BC = 3$ . Find the area of the shaded region (in unit square).

- (A) 4                      (B) 5                      (C) 6                      (D) 7                      (E) 8

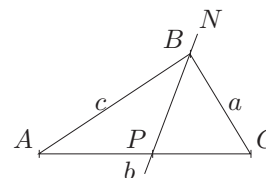


12. Which one of the following is the remainder when  $x + x^7 + x^{16} + x^{37}$  is divided by  $x^4 - x$ ?

- (A) 4                      (B)  $1 + x^2$                       (C) 4x                      (D)  $x + x^3$                       (E)  $x + 3x^2$

13. The triangle  $ABC$  has sides  $a = 13$ ,  $b = 14$ , and  $c = 15$  as shown at the right. Line  $N$  bisects angle  $B$  and crosses side  $b$  at  $P$ . Find  $AP$ , the distance from  $A$  to  $P$ .

- (A)  $3\sqrt{7}$                       (B) 7                      (C)  $7\sqrt{2}$                       (D) 7.5                      (E)  $2\sqrt{15}$



14. Find the sum of the infinite series  $\frac{1}{2} - \frac{2}{4} + \frac{3}{8} - \frac{4}{16} + \frac{5}{32} - \frac{6}{64} + \dots$ .

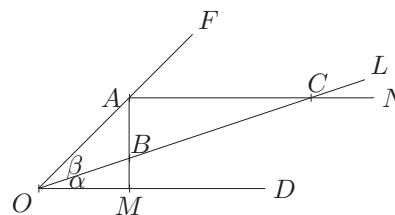
- (A)  $-\frac{1}{3}$                       (B)  $\frac{2}{9}$                       (C)  $-\frac{1}{6}$                       (D)  $\frac{3}{10}$                       (E) none of these

15. Find the rectangular form of the polar equation  $r^2 \cos 2\theta = 8$ .

- (A)  $y^2 = 8x^2$                       (B)  $x = 4y^2$                       (C)  $x^2 - y^2 = 8$                       (D)  $x^2 = 4$                       (E)  $x^2 = 8y^2$

16. Consider the angle at  $O$  formed by the line segments  $\overline{OD}$  and  $\overline{OF}$ . Through point  $A$ , draw line segment  $\overline{AM}$  perpendicular to  $\overline{OD}$  and line segment  $\overline{AN}$  parallel to  $\overline{OD}$ . Next, draw line segment  $\overline{OL}$  crossing segment  $\overline{AM}$  at point  $B$  and segment  $\overline{AN}$  at point  $C$  so that the distance  $BC$  is twice distance  $OA$ . Let  $\alpha$  be the angle at  $O$  formed by the segments  $\overline{OD}$  and  $\overline{OL}$ , and let  $\beta$  be the angle at  $O$  formed by the segments  $\overline{OF}$  and  $\overline{OL}$ . Which one of the following is true?

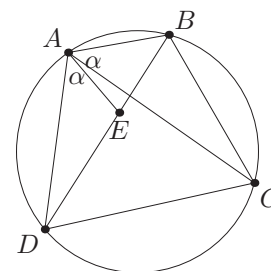
- (A)  $\beta = 2\alpha$                       (B)  $3\alpha^2 = 2\beta^2$                       (C)  $\beta^2 = \alpha$   
 (D)  $\beta = 3\alpha$                       (E)  $3\alpha = 2\beta$



17. The number  $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$  ends with one zero. How many zeros will be at the end of  $50!$  ?  
 (A) 9 (B) 10 (C) 11 (D) 12 (E) 13

18. The polynomial equation  $x^4 - 6x^2 + 56x - 24 = 0$  has two real roots and two complex roots. What is the sum of the two real roots ?  
 (A)  $2\sqrt{6}$  (B) -4 (C) 0 (D) 4 (E)  $-2\sqrt{6}$

19. Let  $ABCD$  be a cyclic quadrilateral (all vertices lie on a circle) with diagonals  $\overline{AC}$  and  $\overline{BD}$  as shown in the sketch. Pick point  $E$  on the diagonal  $\overline{BD}$  so that angle  $DAE$  equals angle  $BAC$ . Which of the following statements are true ?



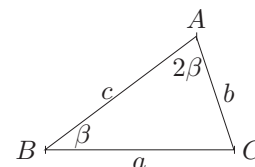
- I. Triangle  $ADE$  is similar to triangle  $ACB$ . II.  $AE = AB$   
 III.  $AE \cdot AC = AB \cdot AD$  IV.  $AB^2 + DC^2 = AD^2 + BC^2$   
 (A) I & II (B) I & III (C) II & IV (D) III & IV (E) I, III & IV

20. If I were to give \$7 to each of the beggars at my door, I would have \$24 left. I lack \$32 of being able to give them \$9 apiece. What is the sum of my money and the number of beggars at my door ?  
 (A) 69 (B) 248 (C) 137 (D) 486 (E) 153

21. Let  $f(x) = ax^2 + bx + c$  be any quadratic with real coefficients  $a, b, c$  having the property that  $|f(x)| \leq 1$  for  $0 \leq x \leq 1$ . Find the smallest number  $M$  such that  $|a| + |b| + |c| \leq M$ .  
 (A) 1 (B) 17 (C) 3 (D) 38 (E) none of these

22. If  $x$  is a real number, then the inverse trigonometric function  $\cot^{-1} x$  is equal to which one of the following ?  
 (A)  $\frac{\pi}{2} - \tan^{-1} x$  (B)  $\frac{1}{\tan^{-1} \frac{1}{x}}$  (C)  $\frac{1}{\tan^{-1} x}$  (D)  $\tan^{-1} \frac{1}{x}$  (E)  $\tan^{-1} \left( \frac{\pi}{2} - x \right)$

23. In triangle  $ABC$  shown at the right, the angle at  $A$  is *twice* the angle at  $B$ . Which one of the following is true ?

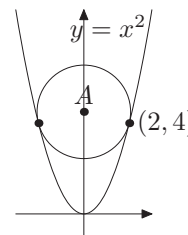


- (A)  $a^2 = b(b + c)$  (B)  $ab + ac = 2bc$   
 (C)  $2a = \sqrt{b^2 + c^2}$  (D)  $\frac{a}{b + c} = \frac{2b}{a + c}$  (E) none of these

24. Each digit of a four-digit personal identification number (PIN) can be any number from 0 to 9, with the restriction that the first digit cannot be 0. Which of the following best approximates the percentage of PIN's that contain at least one digit less than 3 ?  
 (A) 86% (B) 76% (C) 67% (D) 73% (E) 68%

25. A circle with center  $A = (0, a)$  intersects the parabola  $y = x^2$  at  $(2, 4)$  and  $(-2, 4)$ , with all other points on the circle inside the parabola as shown in the figure. Find the number  $a$ .

- (A)  $2\sqrt{5}$       (B)  $\frac{9}{2}$       (C)  $2\sqrt{\frac{13}{2}}$       (D)  $\frac{17}{4}$       (E)  $2\sqrt{8}$



26. The perfect square  $12^2 = 144$  ends with *two* repeated digits 44. It can be proven that the only possible digits that can be repeated at the end of a perfect square are 0's and 4's. What is the **largest possible** number of 4's that can appear at the end of a perfect square?

- (A) 2      (B)  $\boxed{3}$       (C) 5      (D) 13      (E) arbitrarily large

27. Which of the following is the closest approximation to the *circumference* of the ellipse  $7x^2 + 8y^2 = 56$ ?

- (A)  $\sqrt{56}\pi$       (B)  $6\pi$       (C)  $\sqrt{35}\pi$       (D)  $4\pi$       (E)  $\boxed{\sqrt{30}\pi}$

28. Solve  $-3 \leq 2x + 1 \leq 7$ .

- (A)  $-3 \leq x \leq 6$       (B)  $-\frac{5}{2} \leq x \leq \frac{5}{2}$       (C)  $-\frac{1}{2} \leq x \leq \frac{5}{2}$       (D)  $\boxed{-2 \leq x \leq 3}$       (E) none of these

29. Which of the following is equal to  $4 \cos^3 x - 3 \cos x$ ?

- (A)  $\cos 7x \cos x$       (B)  $\sin 7x + \cos x$       (C)  $\boxed{\cos 3x}$       (D)  $\cos 7x - \sin x$       (E) none of these

30. The sequence of Euler numbers  $E_0, E_1, E_2, E_3, \dots$  is defined by

$$E_0 = 1 \quad \text{and} \quad E_n = \sum_{i=1}^n \frac{(-1)^{i+1} (2n)!}{(2n-2i)! (2i)!} E_{n-i}, \quad n = 1, 2, 3, \dots$$

Calculate the number  $E_3$ .

- (A) 3      (B) 15      (C) 48      (D)  $\boxed{61}$       (E) 75

31. A band of 17 pirates stole a sack of gold coins. When they tried to divide the fortune into equal portions, 3 coins remained. In the ensuing brawl over who should get the extra coins, one pirate was killed. The wealth was redistributed, but this time an equal division left 10 coins. Again an argument developed in which another pirate was killed. But now the total fortune was evenly distributed among the survivors. If  $N$  is the least number of coins that could have been stolen, what is the sum of the digits in  $N$ ?

- (A) 9      (B)  $\boxed{12}$       (C) 15      (D) 18      (E) none of these

32. The equation of an ellipse in polar coordinates is given by  $r = \frac{10}{3-2\cos\theta}$ . Which one of the following is the polar coordinates of one of the foci of the ellipse?

- (A)  $\boxed{(0, 0)}$       (B)  $(\frac{10}{3}, \pi)$       (C)  $(4, 0)$       (D)  $(2, \pi)$       (E) none of these

33. A fair die bearing the numbers 1, 2, 3, 4, 5, 6 on its faces is thrown repeatedly until the running total  $T$  first exceeds 12. What is the probability that  $T$  will be 13?

(A)  $\frac{5}{36}$       (B)  $\frac{1}{6}$       (C)  $\frac{5}{21}$       (D)  $\boxed{\frac{2}{7}}$       (E) none of these

34. Find the greatest integer  $n$  for which there exists a simultaneous solution  $x$  to the inequalities

$$k < x^k < k + 1, \quad k = 1, 2, 3, \dots, n.$$

(A) 2      (B)  $\boxed{4}$       (C) 6      (D) 8      (E) none of these

35. The fraction  $\frac{2}{7}$  can be written in a unique way as the sum of two unit fractions  $\frac{1}{a} + \frac{1}{b}$ , where  $a$ , and  $b$  are positive integers, with  $a < b$ . Find  $a + b$ .

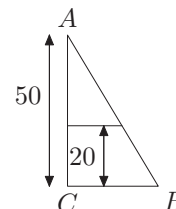
(A) 21      (B) 28      (C)  $\boxed{32}$       (D) 48      (E) 64

36. A rectangle with length  $x$  has area 3 and  $x^3d = 20$ , where  $d$  is the length of the diagonal of the rectangle. Find  $d$ .

(A) 4      (B) 3.5      (C) 2.75      (D)  $\boxed{2.5}$       (E) 2.25

37. The right triangle  $ABC$  has altitude 50. A line parallel to the base has been drawn 20 units above the base to form a right trapezoid with area 320. What is  $CB$ , the length of the base of the right triangle?

(A) 10      (B) 14      (C)  $\boxed{20}$       (D) 26      (E) 32



38. Find the sum of all the solutions to the equation  $2 \log x - \log(2x - 75) = 2$ .

(A) no solutions      (B) 30      (C) 350      (D) 75      (E)  $\boxed{200}$

39. The set  $S = \{13! + 2, 13! + 3, 13! + 4, \dots, 13! + 11\}$  contains ten consecutive integers. How many primes are in the set  $S$ ?

(A)  $\boxed{0}$       (B) 7      (C) 4      (D) 3      (E) none of these

40. Determine the radius of the circle given by  $x^2 + y^2 - 6x + 8y = 56$ .

(A) 6      (B)  $4\sqrt{14}$       (C) 3.5      (D)  $\boxed{9}$       (E)  $5\sqrt{2}$

41. Consider a right triangle with legs of length  $a$  and  $b$  and hypotenuse of length  $c$ . If  $K$  denotes the area of the triangle, which one of the following is equal to  $a + b$ ?

(A)  $\frac{4K}{c^2}$       (B)  $4Kc^2$       (C)  $\frac{c^2}{4K}$       (D)  $c + 2\sqrt{K}$       (E)  $\boxed{\sqrt{c^2 + 4K}}$

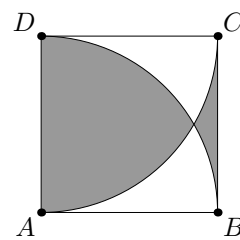
42. How many points  $(x, y)$  with integer coordinates satisfy the inequality  $x^2 + y^2 \leq 25$ ?

(A) 63      (B) 74      (C)  $\boxed{81}$       (D) 99      (E) none of these

43. If  $X + Y = 2N^4 + 2M^4$  and  $XY = N^8 + M^8$ , which one of the following is an expression for  $X$ ?
- (A)  $N^4 - N^2M^2 + 1$  (B)  $N^6M^2 - 2N^4M^4 + N^2M^6$   
 (C)  $N^4 + M^4 - NM$  (D)  $N^6M^2 + 2N^4M^4 + N^2M^6$  (E) none of these

44. If  $AB = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$  is the matrix product of two  $2 \times 2$  matrices where  $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ , what is  $(BA)^{-1}$ ?
- (A)  $\begin{bmatrix} 4 & 9 \\ -1 & -2 \end{bmatrix}$  (B)  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  (C)  $\begin{bmatrix} 4 & -3 \\ 7 & -5 \end{bmatrix}$  (D)  $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$  (E)  $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

45. In the figure shown,  $ABCD$  is a square.  $AB = 1$ ,  $\widehat{AC}$  and  $\widehat{BD}$  are arcs with radius 1 and centers at  $D$  and  $A$ , respectively. What is the difference between the areas of the two shaded regions?



- (A)  $\frac{\pi}{2} - 1$  (B)  $1 - \frac{\pi}{4}$  (C)  $\frac{\pi}{3} - 1$  (D)  $1 - \frac{\pi}{6}$  (E)  $\pi - 1$
46. The number  $\sqrt{19 + \sqrt{336}}$  can be written in the form  $\sqrt{a} + \sqrt{b}$ , where  $a$  and  $b$  are whole numbers and  $a > b$ . What is the value of  $a - b$ ?
- (A) 23 (B) 19 (C) 12 (D) 7 (E) 5
47. If  $g\left(\sqrt{\frac{1+x}{1-x}}\right) = 5x$ , find  $g(2)$ .
- (A) -15 (B)  $\frac{10\sqrt{3}}{3}$  (C)  $15\sqrt{-1}$  (D) -4 (E) 3

48. An open top box (no top) is to be constructed with a square base of length  $x$ , and must have a volume of 16 cubic feet. If the material for the base costs \$1 per square foot and the material for the sides cost \$0.25 per square foot, write the cost of the box,  $C(x)$ , as a function of  $x$ .

- (A)  $x^2 + 8x$  (B)  $\frac{4x^3 + 8}{x}$  (C)  $x^2 + 64x$  (D)  $\frac{x^3 + 16}{x}$  (E) none of these
49. The function  $f(x) = \frac{1}{x^2 + 2x + c}$  will have no vertical asymptote when the constant  $c$  is which one of the following?
- (A) -2 (B) -1 (C) 0 (D) 1 (E) 2
50. The functions  $f(x) = x^3$ ,  $g(x) = \sqrt{x}$ ,  $p(x) = x - 4$ , and  $q(x) = 2x$  can be composed using each function **at most once** to form all of the following functions except one. Which one?
- (A)  $(x - 4)^{3/2}$  (B)  $x^3 - 4$  (C)  $\sqrt{8x^3 - 4}$   
 (D)  $8x^3 - 96x^2 + 384x - 512$  (E)  $2x^3 - 24x^2 + 96x - 64$