2008 ALABAMA

STATEWIDE MATHEMATICS CONTEST

First Round : March 29, 2008
Second Round: April 26, 2008 at The University of Alabama

DIVISION I COMPREHENSIVE EXAM

Construction of this test directed
by
Robert Moore, The University of Alabama

INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of difficulty. For each question, choose the best of the five answer choices labeled A, B, C, D, and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered, and 0 points for each wrong answer. (Thus a "perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate one of more of the answer choices as wrong, then it is to your advantage to guess among the remaining choices.

- All variables and constants, except those indicated otherwise, represent real numbers.
- Diagrams are not necessarily to scale.

We use the following geometric notation:

- If A and B are points, then:
  $\overline{AB}$ is the segment between A and B
  $\overleftrightarrow{AB}$ is the line containing A and B
  $\overrightarrow{AB}$ is the ray from A through B
  AB is the distance between A and B
- If A is an angle, then:
  $m\angle A$ is the measure of angle A in degrees
- If A and B are points on a circle, then:
  $\overset{\frown}{AB}$ is the arc between A and B
  $m\overset{\frown}{AB}$ is the measure of $\overset{\frown}{AB}$ in degrees

Editing by Zhijian Wu, The University of Alabama
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What You Can Do With A Mathematics Major

Occupational opportunities

Actuarial and Insurance	Government	Accountant
Computer & Information Sciences	Investment Analyst	Financial Planner
Researcher	Benefits Specialist	Mathematician
Demographers	Computer Programmer	Cartographer
Data Processor	Navigator	Meteorologist
Applications Programmer	Ecologist	Health
Systems Analyst	Biomedical Engineer	Bio-mathematician
Computer Applications Engineer	Operations Analyst	Operations Research
Control Systems Engineer	Control Systems Engineer	Systems Engineer
Numerical Analyst	Teaching	BusinessIndustry
Statistician	Engineering Analyst	Financial Analyst
Technical Writer	Homeland Security	Communications Engineer

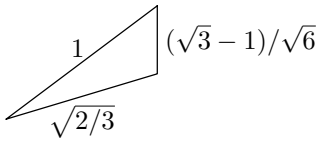
Study in the field of mathematics offers an education with an emphasis on careful problem analysis, precision of thought and expression, and the mathematical skills needed for work in many other areas. Many important problems in government, private industry, health and environmental fields, and the academic world require sophisticated mathematical techniques for their solution. The study of mathematics provides specific analytical and quantitative tools, as well as general problem-solving skills, for dealing with these problems. The University of Alabama offers undergraduate and graduates degrees in Mathematics. Please visit **www.ua.edu** and refer to the undergraduate and graduate programs for additional information.

Engineering Math Advancement Program

The University of Alabama is offering a new summer program to build math skills for students entering engineering. The Engineering Math Advancement Program (EMAP) is a summer residence class that addresses math and engineering prerequisites for incoming engineering students. The program targets bright students who may not have retained the information learned in high school and provides and opportunity to hone technical abilities before entering college. The goal of E-MAP is to assisting entering freshmen in developing a solid background in calculus to succeed in engineering before they start at the University.

Classes are designed around Precalculus Algebra and Trigonometry and incorporate important learning principles to ensure that knowledge is retained and not just memorized. Students develop their skills through hands-on experiences, problem solving teaming exercises, and interaction with engineering professors and instructors through an interdisciplinary Living Laboratory program. Experiments allow students to use simple calculus in engineering applications. The program also involves introducing students to local practicing engineers through work on one or more community service engineering-related activities. E-MAP will reserve 33-40 percent of enrollment space for underrepresented groups. Financial assistance is available based on need. Please visit **emap.ua.edu** for additional information.

1. A triangle has sides $a = 1$, $b = \sqrt{\frac{2}{3}}$ and $c = \frac{\sqrt{3}-1}{\sqrt{6}}$. What is the angle opposite side c?



(**A**) $12°$ (**B**) $\boxed{15°}$ (**C**) $30°$ (**D**) $45°$ (**E**) none of these

2. A cup contains 3 blue balls, 4 red balls and 1 white ball. 4 balls are chosen at random from the cup without replacement. What is the probability that *at least* one red ball is chosen?

(**A**) $\frac{59}{60}$ (**B**) $\frac{34}{35}$ (**C**) $\frac{1}{3}$ (**D**) $\frac{1}{2}$ (**E**) $\boxed{\text{none of these}}$

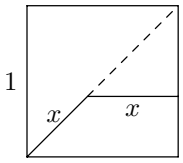
3. The product of three consecutive integers is 2184. What is the sum of the integers?

(**A**) 34 (**B**) 33 (**C**) 36 (**D**) $\boxed{39}$ (**E**) none of these

4. Solve for the positive root of the equation: $(\sqrt{200}+\sqrt{56})\,x^2 + 10\,x - 2(\sqrt{50}-\sqrt{14}) = 0$.

(**A**) $\frac{\sqrt{26}}{\sqrt{14}}$ (**B**) $\sqrt{200}-\sqrt{56}$ (**C**) $\boxed{\frac{5\sqrt{2}-\sqrt{14}}{9}}$ (**D**) $\frac{10}{\sqrt{200}-\sqrt{56}}$ (**E**) none of these

5. The large square in the picture is 1 inch on a side. The point on the diagonal is equally distant from the bottom left corner and the right side. What is that distance (in inches)?



(**A**) $\boxed{2-\sqrt{2}}$ (**B**) $1+\sqrt{2}$ (**C**) $\sqrt{2}$ (**D**) $\sqrt{2}-1$ (**E**) None of these

6. Evaluate: $\cos 67° \cos 22° + \cos 23° \cos 68°$.

(**A**) $\sqrt{2}$ (**B**) $\cos 89°$ (**C**) 0 (**D**) $\boxed{1/\sqrt{2}}$ (**E**) none of these

7. On planet Tris, which circles the star Poly, the natives use length units of dee's and doe's; one dee equals seven does. They don't use square units to measure area, but triangular units: this works fine, since triangles can tile the plane, and one triangular dee is exactly an equilateral triangle whose side is one dee. How many triangular does are there in a triangular dee?

(**A**) 9 (**B**) 3 (**C**) $3\sqrt{3}$ (**D**) 7 (**E**) $\boxed{\text{none of these}}$

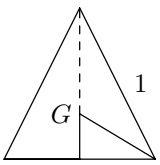
8. a is a real number larger than 1. Simplify the expression: $(2^{-\log_a(a^3)}) \cdot (a^{-\log_a(1/2)})$.

(**A**) $\log_a 2$ (**B**) $\boxed{\frac{1}{4}}$ (**C**) $\frac{1}{2}$ (**D**) $2a$ (**E**) none of these

9. Evaluate $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$, given that an answer exists and is less than e.

(**A**) 4 (**B**) $\sqrt{2}$ (**C**) $\boxed{2}$ (**D**) $2\sqrt{2}$ (**E**) none of these

10. Point G is the centroid of the equilateral triangle. What is the area of the small triangle at the lower right in the picture?



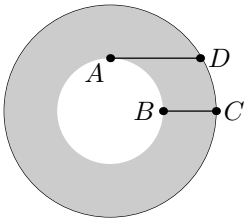
(**A**) $\sqrt{3}$ (**B**) $\sqrt{2}$ (**C**) $\frac{1}{2}\sqrt{3}$ (**D**) $\frac{1}{3}\sqrt{3}$ (**E**) $\boxed{\text{none of these}}$

11. In Professor Wu's latest research paper he defines the concepts of *quenk* and *vetzal* and he proves the Fundamental Theorem of Quenks: **There is no quenk that is not a vetzal.**
A statement that is *completely equivalent* to this theorem is:

(**A**) Every vetzal is a quenk (**B**) Every quenk is a vetzal (**C**) Some vetzals are not quenks

(**D**) Some quenks are not vetzals (**E**) None of these statements is equivalent to Wu's theorem

12. In the figure shown, the two circles are concentric and have radii 1 and 2. $\overline{BC}$, if extended, would pass through the center. $\overline{AD}$ is parallel to $\overline{BC}$ and is tangent to the inner circle. What is the area of the portion $ABCD$ bounded by the two line segments and the two circular arcs?



(**A**) $\frac{3\pi}{8}$ (**B**) $\frac{\sqrt{5}}{2}+\frac{\pi}{12}$ (**C**) $\frac{\pi}{2}-\frac{\sqrt{3}}{3}$ (**D**) $\frac{\sqrt{3}}{2}+\frac{\pi}{12}$ (**E**) none of these

13. Solve the equation:

$$\begin{vmatrix} 1 & 1 \\ 2x & 3x \end{vmatrix} = x$$

where the vertical bars mean the determinant.

(**A**) $x=0$ (**B**) $x=1$ (**C**) $x=0$ or $x=1$

(**D**) Every value of x satisfies this equation (**E**) none of these

14. If x is very close to 1, but not equal to 1, then the expression $\frac{x^2-1}{x-1}$ is very close to what?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) undefined (**E**) none of these

15. A plumber wants to replace an old 6-inch diameter pipe by some newer pipes, which are only 1 inch in diameter. How many of the new pipes will he need to get the same cross-sectional area?

(**A**) 6 (**B**) 6π (**C**) 36π (**D**) 36 (**E**) none of these

16. I am thinking of a two-digit number less than 50. If you double my number and subtract 12, you get the original number with the digits reversed! What is the sum of the digits of the original number?

(**A**) 13 (**B**) 14 (**C**) 15 (**D**) 16 (**E**) none of these

17. A 25 foot ladder is resting against the side of a building, with the foot 7 feet from the base of the building. If the top slips down 4 feet, how far does the bottom of the ladder slip away from the building?

(**A**) 4 (**B**) 5 (**C**) 9 (**D**) 8 (**E**) none of these

18. A triangle has sides of length 24, 10, and 26. What is the radius of the inscribed circle?

(**A**) 26 (**B**) 4 (**C**) 3 (**D**) 8 (**E**) none of these

19. There is a group of boys and girls standing in line at the bus stop. 15 girls get on the first bus to arrive; now there are 2 boys for each girl still at the bus stop. On the second bus, 45 boys get on; now, at the bus stop, there are 5 girls for each boy. What was the original number of girls?

(**A**) 40 (**B**) 43 (**C**) 29 (**D**) 50 (**E**) none of these

20. The price of blue socks is twice the price of black socks. If John buys 20 pairs of black socks and 30 pairs of blue socks he pays \$120. How much does one pair of blue socks cost?

(**A**) \$1.50 (**B**) \$3.00 (**C**) \$4.50 (**D**) \$1.00 (**E**) none of these

21. If $\tan\phi = \frac{4}{3}$, then what is $\tan 3\phi$?

(**A**) $-\frac{44}{117}$ (**B**) $\frac{171}{172}$ (**C**) $\frac{24}{25}$ (**D**) $\frac{25}{24}$ (**E**) none of these

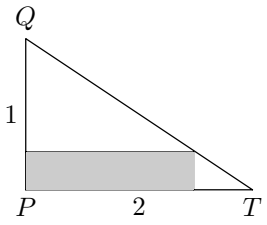
22. You are given the equation $C = \frac{en}{R+nr}$, where e, R and r represent positive constants. Suppose that n starts with value 0 and increases. Then the value of C:

(**A**) increases always (**B**) decreases always (**C**) first increases, then decreases

(**D**) first decreases, then increases (**E**) remains constant

23. The rectangle shown is inscribed in the right triangle with sides lengths 1 and 2. What is the largest possible area of the rectangle?

(**A**) 1 (**B**) 1/2 (**C**) 1/3 (**D**) 1/4 (**E**) none of these



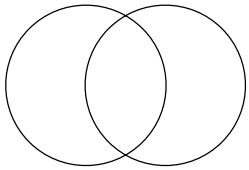
24. Line A is measured to be 10 inches long, with a possible relative error of $\pm 0.2\%$. Line B is measured to be 100 inches long, with a possible absolute error of ± 0.2 inches. Compared to the absolute error in the measurement of line A, the absolute error in line B is

(**A**) greater by 0.18 inches (**B**) greater by 1.8 inches

(**C**) a hundred times greater (**D**) exactly the same (**E**) none of these

25. Two circles of radius 1 are drawn so that each circle passes through the center of the other. What is the *perimeter* of the region of overlap?

(**A**) $\pi/3$ (**B**) $2\pi/3$ (**C**) $4\pi/3$ (**D**) 4/3 (**E**) none of these



26. What is the last digit of 2^{2008}?

(**A**) 2 (**B**) 4 (**C**) 6 (**D**) 8 (**E**) none of these

27. A convex polygon has 100 sides. How many different *diagonals* can be drawn in this figure?

(**A**) 4850 (**B**) 4950 (**C**) 9900 (**D**) 8800 (**E**) none of these

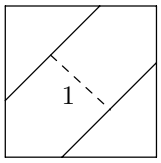
28. Start with an equilateral triangle of side 3. Form a new triangle by joining the midpoints of the sides. Then form a third triangle by joining the midpoints of the sides of the second triangle. Continue in this fashion. The sum of the perimeters of all the triangles is:

(**A**) infinite (**B**) 6 (**C**) 13.5 (**D**) 15.75 (**E**) none of these

29. At 2 : 15 pm, the hands of a clock form an angle of:

(**A**) 30° (**B**) 5° (**C**) $\boxed{22.5^\circ}$ (**D**) 7.5° (**E**) none of these

30. A square has side 2. A stripe of width 1 is drawn inside the rectangle, centered on the diagonal of the rectangle. What is the *area* of the stripe?



(**A**) $\frac{1}{2}$ (**B**) $\boxed{-\frac{1}{2}+2\sqrt{2}}$ (**C**) $\frac{9}{2}-2\sqrt{2}$ (**D**) $\frac{3}{2}\sqrt{2}$ (**E**) none of these

31. A rectangle has sides a and b, with $0 < a < b$. Oddly enough, if you cut the rectangle in half (the cut is perpendicular to one of the sides) then you get two rectangles, each of which is similar to the original rectangle. What is the value of the ratio $\frac{b}{a}$?

(**A**) 2 (**B**) $\frac{1}{\sqrt{2}}$ (**C**) $\frac{1}{2}$ (**D**) $\boxed{\sqrt{2}}$ (**E**) none of these

32. A circular plot of grass 12 feet in diameter is divided by a straight gravel path 3 feet wide. *One edge* of the gravel path passes through the center of the grass plot. What is the area of the *grass* in the plot (in square feet)?

(**A**) $36\pi - 34$ (**B**) $35\pi - 9\sqrt{3}$ (**C**) $30\pi - 15$ (**D**) $36\pi - 33$ (**E**) $\boxed{\text{none of these}}$

33. If $i^2 = -1$, what is the value of $(1+i)^{16} - (1-i)^{16}$?

(**A**) $\boxed{0}$ (**B**) 1 (**C**) 2^8 (**D**) -2^8 (**E**) none of these

34. A sector of central angle θ is constructed in a circle of radius 6. What is the radius of the circle that is circumscribed *about the sector*?

(**A**) $3\cos\theta$ (**B**) $3\sec\theta$ (**C**) $3\cos(\theta/2)$ (**D**) $\boxed{3\sec(\theta/2)}$ (**E**) none of these

35. Which is the largest of the following numbers?

(**A**) $\boxed{3^{6\cdot 9}}$ (**B**) $6^{3\cdot 9}$ (**C**) $9^{6\cdot 3}$ (**D**) $(3\cdot 6)^9$ (**E**) none of these

36. Three girls and three boys are seated around a circular table with six seats. They must alternate: boy, girl, boy, girl, etc. Ignoring rotations, how many different ways are there to seat the people? (Two configurations are different if there are two people who are neighbors in one configuration but not the other.)

(**A**) 3 (**B**) $\boxed{6}$ (**C**) 36 (**D**) 12 (**E**) none of these

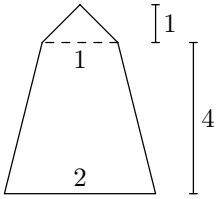
37. 16 college students are going to the beach in 4 identical vans. Each van can hold exactly four students. How many ways can we distribute the students in the vans? (Two distributions are different if there are two students who ride together in one distribution but not in the other. If the same groups of students are together, it does not matter which van they ride in.)

(**A**) 16! (**B**) $(4!)^3$ (**C**) $\binom{16}{4}^3$ (**D**) $\boxed{\binom{16}{4}\binom{12}{4}\binom{8}{4}}$ (**E**) none of these

38. Assume that you are on the equator; the sun passes directly overhead at noon it takes exactly 12 hours to cross from east to west. At 10 : 00 a.m. you measure the shadow of an ancient obelisk to be 10 meters long. How tall is the obelisk ?

(**A**) 20 m (**B**) $\frac{10}{\sqrt{3}}$ m (**C**) $\frac{20}{\sqrt{3}}$ m (**D**) $\boxed{10\sqrt{3}\text{ m}}$ (**E**) none of these

39. Speaking of obelisks, what is the area of the figure below in square units ?



(**A**) 5.5 (**B**) 6.0 (**C**) 6.5 (**D**) 7.0 (**E**) $\boxed{\text{none of these}}$

40. In a race over d yards, A would beat B by 20 yards; B would beat C by 10 yards; and A would beat C by 28 yards. What is the value of d ?

(**A**) $\boxed{100\text{ yd}}$ (**B**) 58 yd (**C**) 116 yd

(**D**) not enough information to answer the question (**E**) none of these

41. For points (x, y) on the line $2x + 3y = 6$, what is the *smallest value* of the quantity $\sqrt{x^2 + y^2}$?

(**A**) $\boxed{\frac{6\sqrt{13}}{13}}$ (**B**) 6 (**C**) $\frac{1}{2}\sqrt{13}$ (**D**) $\sqrt{13}$ (**E**) none of these

42. Five real numbers are chosen and put in order from smallest to largest. The average of all five is 14 . The average of the three middle numbers is only 13 . What is the average of the largest and smallest ?

(**A**) 15 (**B**) $\boxed{15\frac{1}{2}}$ (**C**) 16 (**D**) $16\frac{1}{2}$ (**E**) none of these

43. a is a positive real number and e is the base for the natural logarithm. Simplify the expression: $\log_a(b^{e\cdot\ln a})$.

(**A**) $\frac{e}{\ln b}$ (**B**) $e \cdot \frac{\ln b}{\ln a}$ (**C**) $e \cdot \frac{\ln a}{\ln b}$ (**D**) $\left(\frac{\ln b}{\ln a}\right)^e$ (**E**) $\boxed{\text{none of these}}$

44. A right circular cone and a right circular cylinder have the same height and the same base. Compared to the cone, how large is the volume of the cylinder ?

(**A**) 4 times as large as the volume of the cone (**B**) 2 times as large as the volume of the cone

(**C**) 3/2 times the volume of the cone (**D**) $\sqrt{3}$ times the volume of the cone (**E**) $\boxed{\text{none of these}}$

45. The following is a cryptarithm: each letter stands for a digit (0 through 9), no digit is represented by two different letters, and no number can begin with a 0.

$$SQ \times LQ = RRR$$

What is the value of $S + Q + L + R$?

(**A**) 19 (**B**) 20 (**C**) $\boxed{21}$ (**D**) 22 (**E**) none of these

46. The following is Josh's curriculum in physics and math for the next few years. The number of semesters needed to complete each course is listed, along with the prerequisites:

	Course	Number of Semesters	Prerequisites
BP :	Basic Physics	3	*none*
C :	Calculus	2	*none*
DE :	Differential Equations	1	C
M :	Mechanics	2	C and DE
E :	Electromagnetism	1	BP and C and DE

He is allowed to take up to 5 courses during any semester but he can't enroll in a course without having completed all terms of the prerequisites. What is the *minimum* number of semesters that it will take him to finish this program?

(**A**) 9 (**B**) 6 (**C**) $\boxed{5}$ (**D**) 4 (**E**) none of these

47. A U.S. dime is 11/16 inch in diameter. A nickel is 13/16 inch in diameter. What is the maximum number of dimes that can be laid on a flat surface around a nickel, all tangent to the nickel (and with possibly one small gap between two dimes)?

(**A**) 5 (**B**) $\boxed{6}$ (**C**) 7 (**D**) 8 (**E**) none of these

48. We just blew some air into a spherical balloon and doubled its volume. By how much did we multiply the *surface area*?

(**A**) 2 (**B**) $2^{1/3}$ (**C**) $\boxed{2^{2/3}}$ (**D**) 2^3 (**E**) none of these

49. Let f be a function that maps the interval $(0, \infty)$ into itself. We will say that f is "damping" if, for all $x > y > 0$, we have

$$\frac{x}{y} > \frac{f(x)}{f(y)}$$

Which of the following must be true, for any such function?

(**A**) $\ln f(x)$ is increasing (**B**) $\ln f(x)$ is decreasing (**C**) $\boxed{\ln x - \ln f(x) \text{ is increasing}}$

(**D**) $\ln x - \ln f(x)$ is decreasing (**E**) There are no such functions

50. I have Q quarters and D dimes in my pocket, whose total value is \$2.10. Which of the following equations must be true?

(**A**) $Q + D = 210$ (**B**) $Q + D = 21$ (**C**) $10Q + 25D = 210$

(**D**) $25Q + 10D = 21$ (**E**) $\boxed{\text{none of these}}$