$2^a \cdot 3^b \cdot 5^c \cdot 7^d$

9, -2, 2

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Problems and Solutions

(1) Which of the following numbers is the smallest one?

A: $(2+0+0+9)^{(2+0+0+9)}$
B: $2^9 + 0^0 + 0^0 + 9^2$
C: $\boxed{(((2009^2)^0)^0)^9}$
D: $(20+0+9)^{2009}$
E: 2009

Solution. All five numbers are positive integers. The one under C (and no other one) equals 1, therefore it is the smallest one.

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(2) A handful of coins lie on a table. The number of quarters exceeds by one each of: the number of dimes, the number of nickels, and the number of pennies. What is the smallest possible amount of money on the table, if at least 10 of the coins are identical?

A: $\boxed{\$3.94}$ B: \$4.10 C: \$4.35 D: \$5.00 E: other

Solution. Ten quarters, nine dimes, nine nickels and nine pennies make the minimum amount, which is \$3.94.

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(3) Find the middle digit of the largest three-digit number in which the digit at the units is at least as large as the sum of the digits at the tens and at the hundreds.

A: $\boxed{0}$ B: 1 C: 2 D: 8 E: 9

Solution. 909 satisfies the given condition and clearly is the largest such number. Thus 0 is the correct answer.

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(4) What is the angle (in degrees) formed by the hour and minute hands of a perfect clock at 2:20 p.m.?

A: 30° B: 40° C: $\boxed{50°}$ D: 60° E: 120°

Solution. The minute hand sweeps form 120° in 20 minutes from its vertical position. The hour-hand sweeps $360°/12 = 30°$ in 60 minutes. One-third of it (that is, 10°) will be swept in 20 minutes. So the hour hand will form $60° + 10° = 70°$ with the vertical upward direction. Thus the angle (in degrees) formed by the hour and minute hands 50°.

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(5) Determine the number of different values represented by the expression $\pm 1 \pm 2 \pm 3 \pm 4 \pm 5$.

A: $\boxed{16}$ B: 30 C: 31 D: 32 E: other

Solution. We show that by varying the signs one can obtain every odd integer between $-1-2-3-4-5=-15$ and $1+2+3+4+5=15$. Notice first that the expression $\pm 1 \pm 2$ generates the integers $-3,-1,1,3$. If we add or subtract 3, then these four numbers are shifted up or down on the number line by 3, thus generating every even integer between -6 and 6. Our claim follows by repeating the same argument two more times. After that, observe that the number of all odd integers between -15 and $+15$ is 16, and this is the answer.

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(6) The difference between certain two prime numbers is 2009. How many divisors does the sum of the two primes have?

A: 5 B: $\boxed{8}$ C: 12 D: no such primes E: answer not unique

Solution. Every prime number except 2 is odd. Since the difference between the two prime numbers is odd, the smaller one must be 2. This implies that the larger one must be 2011, which happens to be a prime number. The sum of these two numbers is $2013 = 3 \cdot 11 \cdot 61$. Therefore this sum has eight divisors, namely 1, 3, 11, 33, 61, 183, 671, and 2013.

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(7) A rectangular box has three faces of areas 4.5, 6 and 12. What is the volume of the box?

A: 24.5 B: $\boxed{18}$ C: 49 D: 24 E: other

Solution. Let l, w and h denote the length, width and height of the box, respectively. Then the three given areas are lw, wh and lh (though not necessarily in that order). Multiplying the three quantities we get $lwh^2 = 4.5 \cdot 6 \cdot 12$. Thus the volume of the box is $\sqrt{4.5 \cdot 6 \cdot 12} = 18$.

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(8) Inserting pairs of parentheses into the expression

$$1 \div 1 \div 2 \div 2 \div 3 \div 3 \div 5 \div 5 \div 7 \div 7$$

one can get various rational numbers. How many different integers are among these numbers? (Remark: If no parentheses are inserted, the division operations are performed in the same order as written, from left to right.)

A: 1 B: 4 C: 8 D: $\boxed{16}$ E: 32

Solution. First notice that no matter how the parentheses are inserted into the expression $1 \div 1 \div a \div b \div c \div \ldots$, after simplification the result is a single fraction, where the numerator is the product of some of the numbers $a, b, c, \ldots$ and the denominator is the product of the remaining numbers. The first 1 always ends up in the numerator, while the second 1 always ends up in the denominator. It is also easy to see that we can place the parentheses strategically so that numbers $a, b, c, \ldots$ will end up the numerator or denominator as we choose. The strategy is best explained by an example. Underline the numbers we want in the numerator, say: $1 \div 1 \div \underline{a} \div b \div c \div \underline{d} \div \underline{e} \div \underline{f} \div g$. Each block of consecutive underlined numbers is preceded with a none underlined number. Put each block of consecutive underlined numbers and its preceding number between nested pairs of parentheses according to the following pattern: $1 \div (1 \div \underline{a}) \div b \div (((c \div \underline{d}) \div \underline{e}) \div \underline{f}) \div g$. This expression has the desired simplified form. In our problem, where the numbers $a, b, c, \ldots$ are $2, 2, 3, 3, 5, 5, 7, 7$, the resulting number is an integer if and only if among the integers that appear in the numerator there is at least one 2, at least one 3, at least one 5, and at least one 7. The remaining 2, 3, 5, and 7 can be either in the numerator or in the denominator, independently from each other. This results in $2^4 = 16$ possible different integers.

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(9) During a camping trip there were seven days when it rained either in the morning or in the afternoon, but it never rained in a morning *and* in the afternoon of the same day. There were exactly five dry mornings and six dry afternoons. How many days did the trip last?

A: 7 B: 8 C: **9** D: 10 E: 11

Solution. Assume

- there were $\underline{m}$ days with rain in the morning (and not in the afternoon);
- there were $\underline{a}$ days with rain in the afternoon (and not in the morning);
- there were $\underline{n}$ days with no rain at all, neither in the morning nor in the afternoon;

According to the given information we have

$m + a = 7$, $a + n = 5$ and $m + n = 6$

The sum of these three equations is $2(m + a + n) = 18$. Thus the trip lasted nine days.

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(10) An arithmetic progression has the property that the sum of the first n terms is equal to the square of $2n$. What is the common difference between the consecutive terms of the progression?

A: 4 B: 6 C: **8** D: 10 E: not uniquely defined

Solution. We can find the answer without referring to the formula for the sum of the first n terms. For $n = 1$ we have $a_1 = (2 \cdot 1)^2$, thus $a_1 = 4$. If $n = 2$, then $4 + 4 + d = 4^2$, and this yields $d = 8$. It is easy to see that with these values of a_1 and d, the assumed property holds for the sum of the first n terms: notice that $a_n = 4 + 8(n-1) = 8n - 4$, and thus the sum S_n equals $\frac{1}{2}(a_1 + a_n)n = \frac{1}{2}(4 + 8n - 4)n = (2n)^2$.

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(11) If your expected income in a three-month period in thousands of dollars should be seven more than one-half of your age (in years), then what should your age be when you expect to earn $20,000 in three months?

A: 17 B: 20 C: 22 D: **26** E: 27

Solution. Your income (in a three month period in thousands of dollars) in terms of your age (x) should be $\frac{x}{2} + 7$. We need to solve the equation $\frac{x}{2} + 7 = 20$. The answer is $x = 26$ as the desired age.

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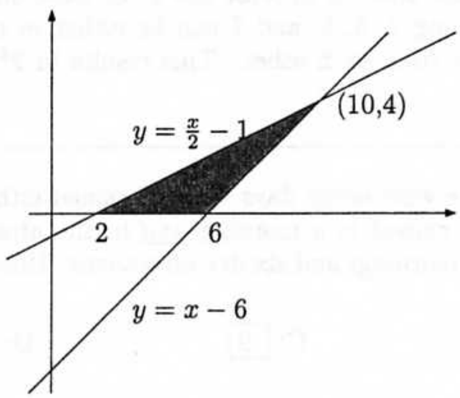
(12) Let $x - 2y - 2 = 0$, $x - y - 6 = 0$ and $y = 0$ be the equations of the lines of the sides of a given triangle. What is the area of the triangle?

A: 4 B: 5 C: 6 D: 7 E: [8]

Solution. After finding the coordinates of the intersection points determined by the three lines it is easy to see that the area is 8.



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(13) At most how many intersection points can 10 lines have if five of them are parallel?

A: [35] B: 45 C: 55 D: 65 E: 100

Solution. Each of the five parallel lines could be intersected by the remaining five lines, creating at most 25 such intersection points. In addition, the 5 remaining lines could be mutually intersecting each other, determining at most 10 more intersection points. Thus we can have at most 35 intersection points, and it is easy to see that this number of intersection points can be reached.

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(14) Peter's and John's average age is three more than the age of Peter. Peter's, John's and Steven's average age is also three more than the average age of Peter's and Steven's. What is the age difference between Peter and Steven?

A: 3 B: [6] C: 9 D: 10 E: 12

Solution. Assume Peter's age is P, John's age is J and Steven's age is S. The following equations express the given information:

$$(1) \qquad \frac{P+J}{2} - 3 = P;$$

$$(2) \qquad \frac{P+J+S}{3} - 3 = \frac{P+S}{2}.$$

From (1) we get

$$(3) \qquad J - 6 = P.$$

From (2) we get

$$(4) \qquad -P + 2J - 18 = S.$$

Using (3) we get

$$(5) \qquad P - S = 6,$$

which answers the question.

(15) A positive integer is called uniformly increasing if its digits reading from left to right are consecutive positive integers. What is the number of all uniformly increasing integers with at least two digits and smaller than 2009?

A: 8 B: 9 C: 16 D: 17 E: 18

Solution. There are 8 two-digit, 7 three-digit, and one four-digit uniformly increasing integers below 2009. Thus the answer is 16.

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(16) Consider the 10-by-10 square S in the xy-coordinate plane, with vertices $(0,0)$, $(0,10)$, $(10,10)$ and $(10,0)$. Let F denote the family of points with integer coordinates and lying on the sides or in the interior of S. What is the length of the shortest path starting at the origin, passing through every point in F, and terminating at some point in F?

A: 40 B: 99 C: 120 D: 121 E: other

Solution. Notice first that set F consists of 121 points, and they can be connected by a spiraling inwards, polygonal path of length 120. This length is a smallest possible, since the distance between any two of the points is at least 1. (Remark. Besides the spiraling one, here are many other paths of the same, minimum length.)

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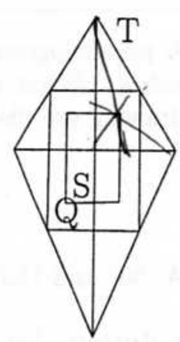
(17) Let T be a deltoid with sides of lengths 4, 4, 6 and 6. Let S be the rectangle formed by the centers of mass of the four right triangles into which the diagonals partition the deltoid. What is the ratio of the areas of the quadrilateral S and T?

A: $\frac{1}{9}$ B: $\frac{2}{9}$ C: $\frac{4}{9}$ D: $\frac{1}{2}$ E: $\frac{5}{9}$

Solution. Denote by Q the rectangle formed by the midpoints of sides of T. It is easy to see that the area of Q is equal to one-half of the area of T. On the other hand, Q and S are similar: Q is obtained by enlarging S from the intersection point of the diagonals by a factor of $\frac{3}{2}$. Thus the ratio of the areas of S and Q is $1 : \frac{9}{4}$. Combining the two ratios we get that the ratio of the areas of the quadrilateral S and T is $1 : \frac{9}{2} = \frac{2}{9}$.



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(18) Find the 2009-th digit after the decimal point in the decimal expansion of $\dfrac{3}{13}$.

A: 0 B: 3 C: 6 D: 7 E: 9

Solution. Use long division to divide 3 by 13. The first six digits after the decimal point are 0.230769 and the remainder after these 6 digits is 3, the same as the number we started with. Consequently, the six-digit block 230769 will repeat. Since the remainder from the division of 2009 by 6 is 5, the 2009-th digit of $\frac{3}{13}$ is the 5-th digit in the group 230769, and the answer is 6.