

Auburn Math-a-thon Ciphering

1.1 The first term of an arithmetic progression is 7, and the tenth term is 187. Find the twentieth term in the progression.

1.2 Find the coordinates of the centroid of a triangle whose vertices have coordinates $(1, 2, 3)$, $(2, 3, 5)$, and $(3, 7, 7)$.

1.3 Find $a + b$ where

$$\begin{aligned}3a + 5b &= 8, \\5a + 8b &= 13.\end{aligned}$$

1.4 Eight students participate in a bowling tournament. Assuming no ties, in how many ways can 1st, 2nd, and 3rd places be awarded?

1.5 Ms. Mancuso has a total of 100 fair six-sided dice, each with sides numbered 1 through 6. She rolls each one of them and adds together the 100 results. Find the probability that the sum is a multiple of 6.

2.1 Johnny Chan is dealt two cards from a standard deck of 52 playing cards. Find the probability that his two cards are of the same rank (a pair such as both aces, both 9's, etc.).

2.2 Let $f(x) = -2x + 7$ and $g(x) = \frac{1}{2}(x + 3)^2$. Find the value of $g(f(17))$.

2.3 Convert 111111_2 to base 10.

2.4 Compute and express in simplest form:

$$\frac{|2 + i|^2}{|2 - i|}.$$

2.5 A radius of a circle is bisected by a chord of length 6. If the radius also bisects the chord, what is the area of the circle?

3.1 Compute the value of

$$\frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)}.$$

3.2 The sum of two real numbers is 16. Find the maximum possible value of their product.

3.3 Evaluate the infinite geometric series

$$1 + \frac{1}{3} + \frac{1}{3^2} + \cdots.$$

3.4 How many of the positive divisors of 972 are odd?

3.5 Let $T_n = 1 + 2 + 3 + \cdots + n$ for each positive integer n . Compute

$$T_1 + T_2 + T_3 + \cdots + T_{12}.$$

4.1 What integer is equal to

$$\sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}$$

4.2 Compute

$$\frac{1}{\log_4 36} + \frac{1}{\log_9 36}.$$

4.3 Poor Jenny was getting dressed for an interview very early one morning when the power went out in her house. The sun had not yet risen, so it was still very dark. Fortunately, she had already picked out all her clothes except socks. She approaches her sock drawer (slowly because it's dark) and reaches inside. She has only four different colors of socks and all of them will match her outfit, but she cannot tell one sock from another. What is the minimum number of socks she must grab in order to know she has a matching pair (same color)?

4.4 Find the sum of the squares of the roots of $3x^2 + 5x + 8$.

4.5 Find the largest positive integer k such that

$$1 + 3 + 5 + \cdots + (2k - 1) < 2008.$$

E.1 Find the smallest positive integer that leaves a remainder of 8 when divided by 9, and a remainder of 10 when divided by 11.

E.2 A cone has a height equal to the diameter of its base. If the slant height of the cone is $6\sqrt{5}$, find the volume of the cone.

E.3 Find the sum of the reciprocals of the solutions to $3x^2 + 31x + 2008 = 0$.

Ciphering Solutions

- 1.1 Let d be the common difference of the sequence (progression). Between the first term and the tenth, there are 9 common differences, so the common difference is

$$\frac{187 - 7}{9} = \frac{180}{9} = 20.$$

To get the twentieth term, we add $20 - 10 = 10$ more common difference to the tenth term:

$$187 + 10(20) = 187 + 200 = \boxed{387}.$$

- 1.2 The coordinates of the centroid of a triangle are the averages of the coordinates of the vertices (see if you can figure out why – hint: the centroid is the point of intersection of the medians).

$$\frac{1}{3}(1 + 2 + 3) = 2,$$

$$\frac{1}{3}(2 + 3 + 7) = 4,$$

$$\frac{1}{3}(3 + 5 + 7) = 5.$$

The coordinates of the centroid are $\boxed{(2, 4, 5)}$.

- 1.3 Notice that the coefficients and constants are all (consecutive) Fibonacci numbers. Subtracting one equation from another produces a similar equation. We continue until we have our answer:

$$\begin{aligned} (5a + 8b) - (3a + 5b) &= 13 - 8 &\Leftrightarrow & 2a + 3b = 5, \\ (3a + 5b) - (2a + 3b) &= 8 - 5 &\Leftrightarrow & a + 2b = 3, \\ (2a + 3b) - (a + 2b) &= 5 - 3 &\Leftrightarrow & a + b = \boxed{2}. \end{aligned}$$

There are other paths to the solution, but this one is insightful.

- 1.4 This kind of counting problem can be most easily thought about in terms of *options*. There are 8 options for choosing the first place finisher. After that, there are 7 options for choosing the second place finisher. After the top two spots are chosen, only 6 bowlers can be chosen for third place. In total, there are $8 \cdot 7 \cdot 6 = \boxed{336}$ ways to award the top three places.

- 1.5 It doesn't matter what Ms. Mancuso rolls with the first 99 dice! Every sum can be expressed in the form $6b + r$ for some integer n and remainder $0 \leq r < 6$. Her last roll brings the sum up to $6(b + 1)$ only when she rolls $6 - r$, and she has a $\boxed{\frac{1}{6}}$ chance of rolling $6 - r$.

- 2.1 Regardless what card Johnny is dealt first, there are 3 cards of that rank remaining in the deck of 51 cards. So, the probability that the second card is of the same rank as the first is

$$\boxed{\frac{1}{17}}.$$

2.2 We typically evaluate composite functions by evaluating the inner function in order to determine the input to the outer function:

$$f(17) = -2(17) + 7 = -34 + 7 = -27.$$

So,

$$g(f(17)) = g(-27) = \frac{1}{2}(-27 + 3)^2 = \frac{1}{2}(-24)^2 = \frac{1}{2} \cdot 576 = \boxed{288}.$$

2.3 We can convert using the value of each digit place:

$$\begin{aligned} 111111_2 &= 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= 1 \cdot 32 + 1 \cdot 16 + 1 \cdot 8 + 1 \cdot 4 + 1 \cdot 2 + 1 \cdot 1 \\ &= 32 + 16 + 8 + 4 + 2 + 1 = \boxed{63}. \end{aligned}$$

Some students might compute $32 + 16 + \dots + 1$ using the formula for the sum of a geometric series:

$$1 + 2^1 + 2^2 + \dots + 2^5 = \frac{2^6 - 1}{2 - 1} = \frac{63}{1} = 63.$$

Yet another solution notes that $1000000_2 - 1_2 = 111111_2$, so

$$111111_2 = 1000000_2 - 1_2 = 1 \cdot 2^6 - 1 \cdot 2^0 = 1 \cdot 64 - 1 \cdot 1 = 64 - 1 = 63.$$

This clever solution requires enough of a familiarity with base numbers to see how to count in a given base.

2.4 This solution can be shortened just a bit by noticing that $|z| = |\bar{z}|$ for all complex numbers z .

$$\begin{aligned} \frac{|2 + i|^2}{|2 - i|^2} &= \frac{(\sqrt{2^2 + 1^2})^2}{\sqrt{2^2 + 1^2}} \\ &= \sqrt{2^2 + 1^2} = \boxed{\sqrt{5}}. \end{aligned}$$

2.5 When the radius gets bisected, the diameter it sits on is cut into pieces of lengths $\frac{r}{2}$ and $\frac{3r}{2}$. The other chord is bisected into pieces of length 3. Then, by power of a point,

$$\frac{r}{2} \cdot \frac{3r}{2} = 3 \cdot 3 \quad \Leftrightarrow \quad \frac{3r^2}{4} = 9,$$

so $r^2 = 12$. Thus, the area of the circle is $\pi r^2 = \boxed{12\pi}$.

3.1 *Solution 1:* There are two ways to proceed with this problem. The given expression is an example of the tangent sum identity, and we can rewrite the expression as

$$\tan\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \tan\frac{7\pi}{12} = \frac{\sin\frac{7\pi}{12}}{\cos\frac{7\pi}{12}}.$$

Students familiar with these sines and cosines can then plug each in and begin to compute:

$$\begin{aligned} \frac{\sin \frac{7\pi}{12}}{\cos \frac{7\pi}{12}} &= \frac{\frac{\sqrt{6}+\sqrt{2}}{4}}{\frac{\sqrt{2}-\sqrt{6}}{4}} \\ &= \frac{\sqrt{6} + \sqrt{2}}{\sqrt{2} - \sqrt{6}} \\ &= \frac{(\sqrt{6} + \sqrt{2})^2}{(\sqrt{2} - \sqrt{6})(\sqrt{2} + \sqrt{6})} \\ &= \frac{8 + 4\sqrt{3}}{-4} = \boxed{-2 - \sqrt{3}}. \end{aligned}$$

Solution 2: Computation is actually a little easier if we just plug in the given tangents and compute:

$$\begin{aligned} \frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{3}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)} &= \frac{1 + \sqrt{3}}{1 - 1 \cdot \sqrt{3}} \\ &= \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \\ &= \frac{(1 + \sqrt{3})^2}{(1 - \sqrt{3})(1 + \sqrt{3})} \\ &= \frac{4 + 2\sqrt{3}}{-2} \boxed{-2 - \sqrt{3}}. \end{aligned}$$

3.2 *Solution 1:* Let x be one of the numbers. Then, the other is $16 - x$, and our goal is to find the maximum of

$$x(16 - x) = -x^2 + 16x.$$

Completing the square helps us get a grip on this quadratic function:

$$-x^2 + 16x = -(x^2 - 16x) = -(x^2 - 16x + 64) + 64 = -(x - 8)^2 + 64.$$

By the Trivial Inequality, the square of a real number must be at least 0, and in this case $(x - 8)^2 = 0$ when $x = 8$, so

$$(x - 8)^2 \geq 0 \quad \Leftrightarrow \quad -(x - 8)^2 \leq 0 \quad \Leftrightarrow \quad -(x - 8)^2 + 64 \leq \boxed{64}.$$

Solution 2: By the Arithmetic Mean-Geometric Mean Inequality (AM-GM), the arithmetic mean of a pair of real numbers is at least as large as their geometric mean. Let a and b be the two numbers in this problem. Then by AM-GM,

$$\frac{a + b}{2} \geq \sqrt{ab}.$$

In this case, $a + b = 16$, so we have that

$$\frac{a + b}{2} = \frac{16}{2} = 8 \geq \sqrt{ab}.$$

Squaring the last inequality, we get $\boxed{64} \geq ab$, with equality (by condition of AM-GM) when $a = b = 8$.

3.3 *Solution 1:* Students who know the formula can get this one quickly:

$$1 + \frac{1}{3} + \frac{1}{3^2} + \cdots = \frac{1}{1 - \frac{1}{3}} = \frac{1}{\frac{2}{3}} = \boxed{\frac{3}{2}}.$$

Solution 2: One of the best ways to work with a complicated series is to give it a name. We let

$$S = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots.$$

The difficult part of working with S is that we only have an infinite expression. We can get a second series that looks a lot like the first by multiplying S by the common ratio of the geometric series, which is $1/3$:

$$\frac{1}{3}S = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots.$$

Now we have two series with almost exactly the same terms. Subtracting the second from the first, we notice that the infinitude of terms mostly cancel and we get

$$\frac{2}{3}S = 1,$$

so $S = \boxed{\frac{3}{2}}$.

3.4 Divisors are best described in terms of their prime factorizations. Note that

$$972 = 2^2 \cdot 3^5.$$

Any divisor of 972 must be in the form $2^a \cdot 3^b$, where a and b are nonnegative integers no greater than 2 and 5 respectively. For an odd divisor, we must have $a = 0$, so these divisors must look like

$$2^0 \cdot 3^b = 1 \cdot 3^b = 3^b,$$

where $0 \leq b \leq 5$. There are 6 possible values of b , which correspond to $\boxed{6}$ odd divisors:

$$3^0 = 1, \quad 3^1 = 3, \quad 3^2 = 9, \quad 3^3 = 27, \quad 3^4 = 81, \quad 3^5 = 243.$$

3.5 *Solution 1:* Compute $T_1 = 1$, then compute each T_n from T_{n-1} . For instance,

$$\begin{aligned} T_2 &= 1 + 2 &&= T_1 + 2 = 1 + 2 = 3, \\ T_3 &= 1 + 2 + 3 &&= T_2 + 3 = 3 + 3 = 6, \\ T_4 &= 1 + 2 + 3 + 4 &&= T_3 + 4 = 6 + 4 = 10, \\ T_5 &= 1 + 2 + 3 + 4 + 5 = T_4 + 5 = 10 + 5 = 15, \\ &&&\vdots \end{aligned}$$

Then add them:

$$1 + 3 + 6 + 10 + \cdots + 78 = \boxed{364}.$$

Solution 2: Viewing the numbers stacked in a grid allows us to use multiplication to simplify the arithmetic:

$$\begin{array}{r}
 1 \\
 1 + 2 \\
 1 + 2 + 3 \\
 1 + 2 + 3 + 4 \\
 1 + 2 + 3 + 4 + 5 \\
 1 + 2 + 3 + 4 + 5 + 6 \\
 1 + 2 + 3 + 4 + 5 + 6 + 7 \\
 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \\
 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \\
 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \\
 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 \\
 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12
 \end{array}$$

Adding down each column, we get

$$12 \cdot 1 + 11 \cdot 2 + 10 \cdot 3 + 9 \cdot 4 + 8 \cdot 5 + 7 \cdot 6 + 6 \cdot 7 + 5 \cdot 8 + 4 \cdot 9 + 3 \cdot 10 + 2 \cdot 11 + 1 \cdot 12.$$

Now we multiply and add:

$$12 + 22 + 30 + 36 + 40 + 42 + 42 + 40 + 36 + 30 + 22 + 12 = \boxed{364}.$$

Solution 3: Students familiar with combinatorics or nice relationships associated with Pascal's Triangle have the quickest approach. We note that

$$T_n = \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1}.$$

By the Hockey-stick Identity,

$$T_n = \binom{n+1}{2}.$$

Now, we use the Hockey-stick Identity again to sum:

$$\begin{aligned}
 T_1 + T_2 + T_3 + \cdots + T_{12} &= \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{13}{2} \\
 &= \binom{14}{3} = \frac{14 \cdot 13 \cdot 12}{3 \cdot 2 \cdot 1} = 14 \cdot 13 \cdot 2 = \boxed{364}.
 \end{aligned}$$

4.1 We begin by giving the expression with infinitely nested radicals a name: N .

$$N = \sqrt{30 + \sqrt{30 + \sqrt{30 + \cdots}}}$$

Using our assigned variable, we substitute for the nested expression within the first radical (because it is the same as the original expression:

$$N = \sqrt{30 + N}.$$

Now, we square our equation to rid ourselves of the radical:

$$N^2 = 30 + N.$$

Now, we organize and factor to solve:

$$N^2 - N - 30 = (N - 6)(N + 5) = 0.$$

Either $N = 6$ or $N = -5$, but N is a radical expression that cannot be negative, so $N = \boxed{6}$.

- 4.2 The ugliest part of the given expression is that the logarithms are in the denominator. We fix this using the identity that switches the base and the argument. The given expression is equal to

$$\log_{36} 4 + \log_{36} 9.$$

Now, we add the logarithms by multiplying the arguments, and the final computation turns out to be the simplest part of the problem:

$$\log_{36} 4 + \log_{36} 9 = \log_{36} 36 = \boxed{1}.$$

- 4.3 The Pigeonhole Principle describes the *worst case scenario* for Jenny (students unfamiliar with this principle are encouraged to look it up!). She has 4 kinds of socks (4 pigeonholes), and the worst case is that she grabs a different color for each of her first 4 tries, giving her 1 of each color (1 pigeon in each pigeonhole). This means she *must* pair one of the first 4 socks when she draws sock $\boxed{5}$. In terms of the pigeonhole principle, the computation is

$$4 \text{ pigeonholes} \times (2 - 1) \text{ pigeons per hole} + 1 \text{ extra pigeon to ensure 2 in } \textit{some} \text{ hole} = \boxed{5}.$$

- 4.4 Let r and s be the roots. Vieta's Formulas give us some good information about how these roots relate to each other:

$$r + s = -\frac{5}{3},$$

$$rs = \frac{8}{3}.$$

Our goal is to compute the sum of the squares, so $r^2 + s^2$. We can get squared terms by squaring the sum of the roots:

$$(r + s)^2 = \left(-\frac{5}{3}\right)^2 \Leftrightarrow r^2 + 2rs + s^2 = \frac{25}{9}.$$

Now, we must subtract out $2rs$ to get what we want:

$$r^2 + s^2 = (r^2 + 2rs + s^2) - 2rs = \frac{25}{9} - 2 \cdot \frac{8}{3} = \boxed{-\frac{23}{9}}.$$

The fact that the sum of the squares of the roots is negative might surprise some students. Were the roots real, this would be impossible. However, the roots of the given quadratic are complex numbers.

4.5 The left-hand side of the inequality simplifies to k^2 . So, we are looking for [the square root of] the largest perfect square less than 2008. We note that

$$1600 < k^2 < 2500 \quad \Leftrightarrow \quad 40^2 < k^2 < 50^2 \quad \Leftrightarrow \quad 40 < k < 50.$$

So, we guess that k is around 45 and note that $45^2 = 2025 > 2008$, but then $44^2 = 1936 < 2008$, so the maximum of k is $\boxed{44}$.

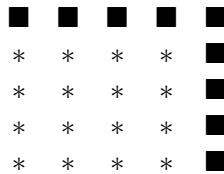
Students wondering how to see that the left-hand side is k^2 can consider any of the following ways to hunt patterns:

Try some small examples:

$$\begin{aligned} 1 &= 1, \\ 1 + 3 &= 4, \\ 1 + 3 + 5 &= 9, \\ 1 + 3 + 5 + 7 &= 16, \\ &\vdots \end{aligned}$$

A pattern is forming!

Think geometrically:



Adding each odd number (like the 9 black squares above) to the previous grid, where there are 4^2 asterisks, achieves a total number of symbols that is the next perfect square: 5^2 total symbols. Adding 11 more symbols around the top and right edges, and we'll have 6^2 . And so on.

Now induct:

$$k^2 + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2.$$

Adding the next odd number to the previous square results in the next perfect square.

E.1 Let a be that integer. Then $a + 1$ is a multiple of both 9 and 11, so it is a multiple of the LCM of 9 and 11, which is 99. So, $a + 1 = 99n$, for some integer n . This means that the smallest possible positive a is $99 \cdot 1 - 1 = \boxed{98}$.

E.2 We make a right triangle out of the height and radius of the cone so that the slant height is the hypotenuse. The height is a leg, equal to $2r$, which is twice the radius, which is the other leg. By the Pythagorean Theorem,

$$r^2 + (2r)^2 = (6\sqrt{5})^2 \quad \Leftrightarrow \quad 5r^2 = 180.$$

So, $r^2 = 36$, and $r = 6$. The area of the base is $6^2\pi = 36\pi$ and the height of the cone is $2 \cdot 6 = 12$, so the the volume is

$$\frac{36\pi \cdot 12}{3} = \boxed{144\pi}.$$

E.3 Let r_1 and r_2 be the solutions of the given equation. Then, by Vieta's Formulas,

$$r_1 + r_2 = -\frac{31}{3},$$

$$r_1 r_2 = \frac{2008}{3}.$$

Now, we note that

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2} = \frac{-\frac{31}{3}}{\frac{2008}{3}} = \boxed{-\frac{31}{2008}}.$$