

Solutions to Individual Test

1. Let the circle have center at the origin and let the rectangle have a side parallel to the x -axis. Then the rectangle meets the circle in the first quadrant at the point $(a, \sqrt{9-a^2})$. So the midpoints of the rectangle in the first quadrant will be at the points $(a, 0)$ and $(0, \sqrt{9-a^2})$. The length of this segment is $\sqrt{a^2 + (\sqrt{9-a^2})^2} = \sqrt{9} = 3$. Thus the perimeter is $4(3) = 12$.
2. $m(\angle ABC) = \frac{180^\circ \cdot 3}{5} = 108^\circ$; $\angle BAC \cong \angle BCA$ so $m(\angle BAC) = \frac{72^\circ}{2} = 36^\circ$.
3. Draw two radii from the center of the circle to two vertices of the triangle. Call these lengths r . We know that the angle between these two radii is 120° . Now, using the Law of Cosines $5^2 = r^2 + r^2 - 2rr \cos(120)$ which gives us $25 = 2r^2 - 2r^2(-1/2) = 3r^2$. Thus $r^2 = 25/3$ or $r = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$. Note: we could also solve this problem using the $30^\circ, 60^\circ, 90^\circ$ triangle method by drawing a line segment from the center of the circle to the midpoint of an edge of the triangle and using the radius as the other side.
4. Simplifying we get $x = \frac{a}{bc}$ and $y = \frac{ac}{b}$. Thus $\frac{x}{y} = \frac{\frac{a}{bc}}{\frac{ac}{b}} = \frac{a}{bc} \frac{b}{ac} = \frac{1}{c^2}$
5. Slice the tetrahedron with a plane from the top which is perpendicular to the base. This gives us an isosceles triangle of base 12 and height 9. Then from the top go down 6 units to get a similar triangle. Using ratios we have $\frac{9}{12} = \frac{6}{\text{base}}$. This gives us a base length of 8 which is the edge length of the square with area 64.
6. Let $x = 0.131929\overline{29}$. Then $100x = 13.1929\overline{29}$. Thus $100x - x = 13.061$. So $99x = 13.061$ or $x = \frac{13061}{99000}$.
7. Just using the Pythagorean Theorem twice we get $\sqrt{\sqrt{5^2} + \sqrt{5^2} + \sqrt{5^2}} = \sqrt{5 + 5 + 5} = \sqrt{15}$.
8. Since $\log_7 5 = A$ then $7^A = 5$. Now we get $\log_5 7^A = \log_5 5 = 1$. So $A \log_5 7 = 1$ which gives us $\log_5 7 = 1/A$.
9. $\log_2(\log_3 x) = 2$ is equivalent to $\log_3 x = 2^2 = 4$. This, in turn, is equivalent to $x = 3^4 = 81$.
10. There are the eleven zeros from 10, 20, 30, 40, 50, 60, 70, 80, 90, and 100. Then there are all the 5s. One each from 5, 15, 35, 45, 50, 55, 65, 85, and 95. And then two 5s from each of 25 and 75. Multiplied by the necessary 2s we get 13 more zeros. Thus we have 24 zeros.
11. Let's start by using 12 o'clock as 0° and proceeding in a clockwise fashion. So the minute hand is at the number 3 which is 90° . The hour hand is at $5 + 1/4$ which is $(5.25/12)360^\circ = 157.5^\circ$. The difference is 67.5° .
12. A square inscribed in an equilateral triangle that does not have an edge lying along the edge of the triangle can be enlarged by rotating it so that a side is parallel to a triangle edge and

then expanded further until the sides are coincident. If the square is not symmetrically placed then moving it to the left or right will enlarge it. So the square we seek is symmetrically placed with an edge coincident with an edge of the triangle.

Let x be the side of the square; calculations are in inches. The small triangle cut off by the square is similar to the 30-60-90 triangle. The side of the triangle has length $\frac{8}{\sqrt{3}}$ and half the side has half that length. Thus by similar triangles:

$$\frac{\frac{4}{\sqrt{3}} - \frac{x}{2}}{x} = \frac{1}{\sqrt{3}}$$

$$\frac{4}{\sqrt{3}} - \frac{x}{2} = \frac{x}{\sqrt{3}}$$

$$\frac{4}{\sqrt{3}} = x \left(\frac{1}{\sqrt{3}} + \frac{1}{2} \right)$$

Thus:

$$x = \frac{4}{\sqrt{3}} \left(\frac{2\sqrt{3}}{2 + \sqrt{3}} \right) = \frac{8}{2 + \sqrt{3}}.$$

13. The plane must cut each face of the cube from the midpoint of one side to the midpoint of an adjacent side forming a right 45° triangle. The side of this triangle is $\frac{3}{2}$ so the hypotenuse of this triangle is the required length and is equal to $\frac{3}{2}\sqrt{2}$.
14. First we argue that an inscribed triangle can be translated or rotated in the square so that they have a common vertex. Then we argue that the side of this triangle (with a common vertex with the square) can be expanded to a "rigid" triangle in the square symmetric about the diagonal (it is important that $60^\circ < 90^\circ$ for this step.) Let s denote the side of the triangle. The side not adjacent to the common vertex cuts the square in a 45° angle with the triangle side along the hypotenuse; so the other side of this right triangle has length $\frac{s}{\sqrt{2}}$. Applying the Pythagorean theorem to the other side yields:

$$s^2 = 3^2 + \left(3 - \frac{s}{\sqrt{2}} \right)^2.$$

Expanding and collecting terms:

$$\frac{s^2}{2} + \frac{6}{\sqrt{2}}s - 18 = 0.$$

Using the quadratic equation after reduction gives (you need to take the plus root):

$$s = 3\sqrt{6} - 3\sqrt{2}.$$

15. Let $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}$. Then $x^2 = 6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}$. Thus $x^2 = 6 + x$, or $x^2 - x - 6 = 0$. Solving this we get $(x - 3)(x + 2) = 0$. So $x = -2, 3$. Clearly $x = -2$ is not a plausible answer. So $x = 3$.
16. Let $x < y < z$ be the radii of the three circles. Each side of the triangle intersects one of the points of tangency. Thus, $x + y = 3$, $x + z = 4$, and $y + z = 5$. Solving these three equations simultaneously gives $x = 1, y = 2, z = 3$. Thus, the largest of the three circles has radius 3.
17. It is obvious that one dollar and three dollars can not be paid using exact change. Obviously, all of the even numbers can be paid using two dollar bills. Also, any odd number greater than or equal to five can be paid by using a five dollar bill plus the appropriate number of two dollar bills. Thus, three dollars is the largest which can not be paid using exact change.
18. It is a well known result of geometry that an inscribed angle in a circle always measures exactly one half of the subtended arc. In this case, the subtended arc is 240 degrees ($2/3$ of the 360 degrees of the circle). Thus, the inscribed angle (which has the same measure regardless of where the point D lies) has a measure of 120 degrees.
19. Observe two facts: (1) $3^{2008} = 81^{502}$, and (2) any power of an integer whose last digit is 1 and with a positive integer in the exponent is again an integer ending with a "one". Thus the answer is 1.
20. Let us write down how many students are after the 1st, 2nd, 3rd, etc. sound of the whistle, and try to find a pattern.

	1st	2nd	3rd	4th	5th	classrooms
After the 1st whistle	999	1	0	0	0	0
After the 2nd whistle	998	2	0	0	0	0
After the 3rd whistle	997	2	1	0	0	0
After the 4th whistle	996	2	2	0	0	0
After the 5th whistle	995	2	2	1	0	0
After the 6th whistle	994	2	2	2	0	0

Initially only one class room is not empty. The number of none empty classrooms is increased by one at each odd whistle sound. Since there are 30 odd whistle sounds during the first hour, the number of non empty class rooms is 31. Therefore the number of empty classrooms after one hour is 969.

21. Notice that $44^{44} = 2^{88}11^{44}$ and $8^n = 2^{3n}$. Now the question is the following: What is the largest n such that $3n \leq 88$? Clearly $n = 29$ is the largest such integer.
22. "12 people complete a certain job in 5 days" means that one person in one day completes $\frac{1}{60}$ -th of the job. If 15 people complete half of the job in n days, then $15n \frac{1}{60} = \frac{1}{2}$, thus $n = 2$.
23. Since the largest common divisor of the two unknown integers is 8, we can write the two integers as $8n$ and $8m$ where n and m are relative primes. The least common multiple of these two integers is $8mn = 48$. The product of the two integers must be $8m \times 8n$, and thus it is $8 \times 48 = 384$. Finally the sum of the digits of 384 is 15.

24. The first and the last digits must be different from zero, thus we have 9 choices for them. We do not have restrictions for the two digits in the middle, so we have 10 choices for each of them. The digits are chosen independently from each other, thus we can create $9 \times 10 \times 10 \times 9 = 8100$ different four digit numbers.
25. $5m - n = 5m - n - (m - 5n) + (m - 5n) = 4m + 4n + 4 = 4(m + n)$. Thus only multiples of 4.
26. Since $10 = 2 \times 5$. the sum of the digits is 9 only if the digits are 1, 1, 2, and 5. If the digits would be different, then we would be able to make $4!$ four digit numbers. Since two of the digits are the same the number of different outcomes is only half of $4!$. Thus there are 12 integers with the given properties.
27. The place values of digits of integers in the base 5 from right to left are: 1, 5, 25, 125, 625, ... Similarly the place values of digits of integers in the base 6 from right to left are: 1, 6, 36, 216, 1296, ... The integers whose representations in the base 5 system consist of exactly four digits are at least 125 and at most 624. Similarly the integers whose representations in the base 6 system consist of exactly four digits are at least 216 and at most 1296. The common part of these two integer intervals starts at 216 and ends with 624, i.e it contains 409 integers.
28. The following table contains the squares of the single digit numbers, and that immediately reduces the possible endings to 11, 44, 55, 66, 99.

b	0	1	2	3	4	5	6	7	8	9
b^2	0	1	4	9	16	25	36	49	64	81

Every integer can be written in the form $10a + b$ where a is an integer and b is a single digit integer. Thus the square of $10a + b$ has the form $100a^2 + 20ab + b^2$. Notice that the term $100a^2$ does not effect the last two digits of the square. Furthermore $20ab$ always has an even digit at the tens. This implies that

if $b = 1$, then at the tens in the square we must have an even digit,

if $b = 5$, then at the tens in the square we must have an even digit,

if $b = 6$, then at the tens in the square we must have an odd digit,

if $b = 9$, then at the tens in the square we must have an even digit.

thus we are left with the case 44, which can be the last two digits of a square, as the example $12^2 = 144$ shows.

29. The square S is partitioned by its diagonals into four right isosceles triangles that can be reassembled so as to form two unit squares. Hence the area of S is 2. The triangle T is right (since the sum of the squares of two of its sides is equal to the square of the third side), and its sides adjacent to the right angle are $3/2$ and 2. Hence the area of T is $\frac{3}{2}$. The area of the hexagon H is six times the area of an equilateral triangle of side $\frac{\sqrt{3}}{2}$, which amounts to $\frac{9}{8}\sqrt{3}$. To compare the area of H with the areas of T and S without a calculator, compare the squares of the areas instead. Thus the answer is $T < H < S$.

30. The selected points are vertices of a regular tetrahedron of edge length $\sqrt{2}$, hence the problem is well-posed (the volume in question is independent from the choice of the four points, and there are two such choices). The faces of this tetrahedron partition the cube into five tetrahedra: one is the tetrahedron $ABCD$ and each of the other four is defined by a face of $ABCD$ and a vertex of the cube separated from the tetrahedron $ABCD$ from the face. The volume of each of the four "cube-corner" tetrahedra is $\frac{1}{6}$, thus the volume of $ABCD$ is
- $$1 - \left(4 \times \frac{1}{6}\right) = \frac{1}{3}.$$