1998 Hoover High School Math Tournament
Comprehensive Examination

1. Tiger Woods shoots consecutive golf scores of 64, 67, 65, and 72. What score must Tiger shoot in a fifth round to lower his average by 1?

a. 60 b. 61 c. 62 d. 63 e. none of these

2. A right triangle that is inscribed in a circle has hypotenuse length 42. Find the area of the circle.

a. 21π b. 42π c. 441π d. 1764π e. none of these

3. Simplify: $2\left[\cos\frac{\pi}{12}+\sin\frac{\pi}{12}\right]$.

a. $\frac{\sqrt{2}}{2}$ b. $\sqrt{2}$ c. $\frac{\sqrt{6}}{2}$ d. $\sqrt{6}$ e. none of these

4. A regular hexagon and a square are each inscribed in the same circle. Find the ratio of the area of the hexagon to the area of the square.

a. $\frac{4\sqrt{3}}{9}$ b. $\frac{3\sqrt{3}}{4}$ c. $\frac{2\sqrt{3}}{9}$ d. $\frac{3\sqrt{3}}{8}$ e. none of these

5. Bill Gates is worth 6.5×10^9 dollars. If there are 250 million people in the United States and Bill Gates is to divide his wealth evenly, then what is the most that he could give each person in the United States?

a. \$26 b. \$260 c. \$25 d. \$250 e. none of these

6. The vertex of the parabola $y=20x^2+80x+60$ is an ordered pair of the form (a,b). Find the value of $a\div b$.

a. -19 b. -20 c. -21 d. -22 e. none of these

7. What is the remainder when 2^{14} is divided by 7?

a. 2 b. 4 c. 5 d. 6 e. none of these

8. Let $f(x)=\cos x$ for all real x. Simplify $\frac{\left[f(-x)-[f(x)]^2\right]\left[f(-x)+[f(x)]^2\right]}{[f(x)]^2}$.

a. $\cos^2 x$ b. 1 c. $\sin^2 x$ d. $\tan^2 x$ e. none of these

9. Given that $\sum_{n=0}^{\infty} r^n = 1998$, find the value of $1998r$.

a. 1 b. 1997 c. 1998 d. $\frac{1997}{1998}$ e. none of these

10. The graph of the equation $r^2 = \frac{48}{\cos^2\theta + 3\cos 2\theta}$ is a(n):

a. hyperbola b. parabola c. ellipse d. circle e. none of these

11. For a given matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define its 3-expansion by the relation:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b & a+b \\ c & d & c+d \\ a+c & b+d & a+d \end{bmatrix}.$$

Find the sum of all values of x such that the determinant of $T\left(\begin{bmatrix} x & 1 \\ 2 & 3 \end{bmatrix}\right)$ is -39.

a. -5 b. -4 c. 4 d. 5 e. none of these

12. Let $f(x) = x^5 + x + 17$, and let g be the inverse of f. Find the value of $g(f(g(15)))$.

a. 15 b. 30 c. -1 d. -15 e. none of these

13. Using only the digits 1, 2, 3, 4, 5, and 6, and using each digit at most once, there are $6 \cdot 5 \cdot 4 \cdot 3 = 360$ four digit numbers that can be formed. Find the average of all such four digit numbers.

a. 2654.5 b. 3888.5 c. 2907.5 d. 3523.5 e. none of these

14. Suppose a, b, c, and d are positive numbers satisfying the following system:

$$\ln\left[(a+b)^2\right] = 16 + 2\ln\left(\frac{1}{c+d}\right)$$

$$2\ln\left(c^3 + d^3 + 3cd^2 + 3c^2d\right) = -15 + 3\ln(a+b)$$

Determine the value of $a+b$.

a. e^7 b. 7 c. e^8 d. 8 e. none of these

15. Let r_1 and r_2 be the roots of the equation $x^2 - sx + t = 0$, where $s > t \geq 1$. Given that $r_1^4 + r_2^4 = 0$, find the exact value of $\frac{s^2}{t}$.

a. $\sqrt{2+\sqrt{2}}$ b. $2-\sqrt{2}$ c. $2 \pm \sqrt{2}$ d. $2-\sqrt{2}$ e. none of these

16. In ΔABC, $BC = a$, $CA = b$, $AB = c$, and $m\angle B = 61°$. Find the value of

$$\frac{b^{1998}\left[\cos(A-C) - \cos 119°\right]^{999}}{2^{999} a^{999} c^{999} \sin^{1998} 61°}.$$

a. $\frac{1}{2^{999}}$ b. 2^{999} c. 1 d. 0 e. none of these

17. How many integers between 50 and 100 inclusive have exactly 4 positive integer divisors?

a. 16 b. 17 c. 18 d. 19 e. none of these

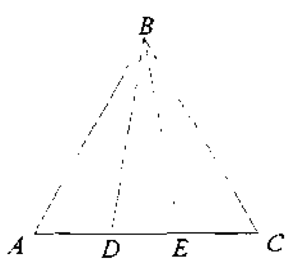
18. Evaluate: $\lim_{x\to\infty} \frac{x}{\sqrt{16x^2 + 1998x + 13}}$.

a. $\frac{1}{4}$ b. 1 c. 0 d. diverges e. none of these

19. How many solutions for $-2\pi < x < 2\pi$ are there to the equation $\cos x = e^x$?

a. 0 b. 1 c. 2 d. 3 e. none of these

20. In equilateral triangle ABC, segments are drawn from point B to points D and E on $\overline{AC}$ such that $\angle ABD = \angle DBE = \angle EBC$. Given that $BC = 6$, find the length of $\overline{EC}$.



a. $6\cos 10° \sec 70°$ b. $6\cos 70° \sec 10°$ c. $6\sin 70° \csc 10°$ d. $6\sin 10° \csc 70°$ e. none of these

21. Consider the following polynomial with *real* coefficients:

$$P(x) = x^9 + a_8 x^8 + a_7 x^7 + a_6 x^6 + \ldots + a_2 x^2 + a_1 x + 5.$$

If the equation $P(x) = 0$ has 9 positive, real solutions, then what is the maximum possible value of a_8?

a. $-9\sqrt[8]{5}$ b. $-9\sqrt[9]{5}$ c. $-8\sqrt[8]{5}$ d. $-8\sqrt[9]{5}$ e. none of these

22. Find the amplitude of the graph of $y = 3\sin x + 7\cos x$.

a. 40 b. $2\sqrt{10}$ c. 58 d. $\sqrt{58}$ e. none of these

23. A sequence, $\{a_n\}$, is defined such that $a_1 = c$ and $a_{n+1} = \frac{n+1}{4n} a_n$ for all $n \geq 1$.

Define $S_n = \sum_{k=1}^{n} a_k$. Find $\lim_{n \to \infty} S_n$.

a. $\frac{16c}{9}$ b. $\frac{4c}{3}$ c. $\frac{9c}{16}$ d. $\frac{3c}{4}$ e. none of these

24. Pentagon $ABCDE$ has a right angle at D and $CD = DE = 8\sqrt{2}$. If $\overline{AB}$ is parallel to $\overline{CE}$, the perpendicular distance from D to $\overline{AB}$ is 20, and the area of $ABCDE$ is K, then compute the minimum possible integral value of K.

a. 154 b. 156 c. 158 d. 160 e. none of these

25. Three real numbers are selected at random between 0 and 1. Given that at least one of the three numbers is less than ½, find the probability that the sum of the three numbers is less than 1.

a. $\frac{7}{12}$ b. $\frac{7}{24}$ c. $\frac{7}{36}$ d. $\frac{7}{48}$ e. none of these

TIEBREAKERS

TB1 Find the smallest positive integer x such that $\sec 70x° = 1 - \frac{\tan 40°}{\tan 20°}$.

TB2 Jack and Jill play a game where the first person to make a basketball shot wins. If the probability that Jack makes a basketball shot is 1/3, the probability that Jill makes a basketball shot is 2/3, and Jack goes first, find the probability that Jill wins.

TB3 Find the sum of the first thirty positive perfect cubes.