

Soli Deo Gloria Fall Tournament Ciphering Round

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Solution.

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1 Instructions

This round consists of 8 problems, each equally valued, given in pairs. You will have 10 minutes to work on each pair and write your answer on the provided answer sheets. It is recommended that you fill out the name and school fields on your answer sheets before the round begins.

2 Set 1

Problem 1. Consider a variety of paper sizes, A_i for $i \in \mathbb{Z}$ and $i \geq 0$. The size A_0 is defined to be 841 x 1189 mm. For each subsequent size, A_{i+1} is formed by dividing A_i in half with a cut parallel to its shortest side. How large is A_4 paper? Write your answer as [number] x [number]. Answers whose dimensions are within 1 unit of correct will be accepted.

Solution. We have A_0 paper of size 841x1189 mm, so A_1 paper is 594.5x841, $A_2=420.5x594.5$, $A_3=297.25x420.5$, $A_4=\boxed{210.25x297.25}$.

Answers in the range (209.25,211.25) x (296.25,298.25) will be accepted. In particular, 210x297 will be accepted. The order does not matter. \square

Problem 2. In base 3 there are 3 digits: 0,1, and 2. Suppose we count in base 3, except we use the digits $\bar{1}$, representing -1, 0, and 1, so that $\bar{1}01$ is equal to $-1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 = -8$; this method of counting is called *balanced ternary*. Jack takes the positive integers in order from 1, represents them in balanced ternary, and writes the result down. What is the 387th digit he writes down?

Solution. Every positive integer in balanced ternary begins with the digit 1: if one started with $\bar{1}$, the remaining digits, even if all 1, would have a value smaller than the $\bar{1}$.

There is 1 1-digit positive balanced ternary integer, $1 \cdot 3 = 3$ 2-digit integers, $1 \cdot 3^2 = 9$ 3-digit integers, 27 4-digit integers, and 81 5-digit integers. He will write down $1 \cdot 1 + 3 \cdot 2 + 9 \cdot 3 + 27 \cdot 4 = 142$ digits before he reaches the 5-digit integers.

In the 5-digit integers beginning in $1\bar{1}$, there are 27 integers, which take up $27 \cdot 5 = 135$ digits. He has 110 digits to go.

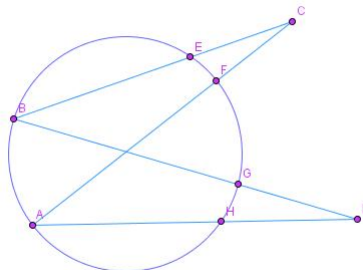
In the 5-digit integers beginning in $10\bar{1}$, there are 9 integers, taking up 45 digits. There are 65 digits to go.

In the 5-digit integers beginning in 100 , there are 9 integers, taking up 45 digits. There are 20 digits to go.

The next integers are $101\bar{1}\bar{1}$, $101\bar{1}0$, $101\bar{1}1$, and 10100 . Thus the last digit he writes will be $\boxed{0}$. \square

Problem 3. If a and b are integers such that $ab = 4014$, what is the probability that $a^{1-b}b^{-1} = 4014^{-1}$?

Solution. Adapted from a problem posted in the Mathcounts forum on [artofproblemsolving.com](http://www.artofproblemsolving.com) by `archimedes1`. Original topic at <http://www.artofproblemsolving.com/Forum/viewtopic.php?t=164346>



For $ab = 4014$ and $a^{1-b}b^{-1} = 4014^{-1}$, we must have $1 - b = -1$ or $a = 1$ or $a = -1$ and b even.

First case: $1 - b = -1 \rightarrow b = 2$, one solution.

Second case: $a = 1$: also one solution.

Third case: $a = -1$, b even: a third solution.

The prime factorization of 4014 is $2 \cdot 3^2 \cdot 223$, so there are 24 pairs of divisors of 4014 (a and b do not have to be positive integers!), so the probability desired

$$\text{is } \frac{3}{24} = \boxed{\frac{1}{8}}.$$

This is an easy problem, but it has tricky cases. □

Problem 4. In the diagram, the measure of $\angle CAD$ is 38 degrees. $4AB = CB$, $4GH = DH$, $AB = 3GH$, and $CA = 3DG$. Find $\angle CBD$.

Solution. Thanks be unto Laura Zehender for this problem.

$$\frac{CA}{DG} = \frac{AB}{GH} = \frac{CB/4}{DH/4} = \frac{CB}{DH}$$

So $\triangle CBA \sim \triangle DHG$.

By Power of a Point, $DG \cdot DA = DH \cdot DB$. So $DG/DH = DB/DA$, and $\triangle DHG \sim \triangle DAB$. Thus $\triangle CBA$ and $\triangle DAB$ are similar, and since $AB = BA$, they are also congruent.

$$\text{So } \angle CBD = \angle DBA - \angle CBA = \angle CAB - \angle DAB = \angle CAD = \boxed{38^\circ} \quad \square$$

Problem 5. Find all pairs (m, n) for which $\binom{m}{n} = 2007$.

Solution. Problem inspired by *360 Problems for Mathematical Contests*, Chapter 2, Problem 18.

WOLOG let $n < \lceil \frac{m}{2} \rceil$.

If $n = 0$, we have $\binom{m}{n} = 1$, no solutions.

If $n = 1$, we have $m = 2007$, two solutions [since $n = 2006$ also works].

If $n = 2$, we have $m(m-1) = 4014$. The only factors of 4014 are 2, 3, 6, 9, 18, 223, ... none of whom are such that $m(m-1) = 4014$. No solutions.

If $n \geq 3$, we have $\binom{m}{n} \geq \binom{m}{3} = \frac{m(m-1)(m-2)}{6}$, so we have $m(m-1)(m-2) \leq 2007 \cdot 6$. By adding -24360 to both sides, we have $(m-30)(m^2 + 27m + 812) \leq -12318$. We must thus have $m \leq 30$. But this means $\frac{(m)(m-1)\dots(m-n+1)}{n!}$ does not contain the factor 223 of 2007.

Thus the only solutions are $\boxed{(2007, 1), (2007, 2006)}$. □

Problem 6. In triangle ABC , cevians AP , BN , and CM intersect at D . Angles AND and APB have equal measure. Segment AB has length 15, MA has length 6, and AD has length 8. Also, the ratio of CA to BC is 5:6. Find BD .

Solution. This problem is also due to Laura Zehender. Blame her! ;-)

By Ceva's Theorem, $\frac{MA}{MB} \times \frac{NC}{NA} \times \frac{PB}{PC} = 1$

Since $\angle AND = \angle BDP$ and $\angle AND = \angle APB$, $\triangle AND \sim \triangle BPD$.

So $\frac{CA}{BC} = \frac{NC}{PC}$

Also $\angle APC = \angle BNC$, and obviously $\angle ABC = \angle ABC$, so $\triangle BNC \sim \triangle BCA$.

Thus $\frac{AD}{BD} = \frac{PB}{NA}$

So $\frac{MA}{MB} \times \frac{CA}{BC} \times \frac{AD}{BD} = 1$

Since $AB = 15$, $MA = 6$, $AD = 8$, and $CA/BC = 10/12$, these can be replaced in the equation, so $\boxed{BD = 10}$. □

Problem 7. Randall is running away from a velociraptor. He's running at 15 m/s. The velociraptor is 50 m away from Randall when she spots Randall, and immediately begins chasing him. She accelerates from rest at a constant 10m/s^2 . In how many seconds does the velociraptor catch up to Randall?

Solution. Problem inspired by an XKCD cartoon, which are written by Randall Munroe. xkcd.com

Suppose the velociraptor runs for time t , reaching a top speed of $10t$ m/s. For time a , $a < t$, he will travel at $10a$ m/s, while at time $t - a$ he will travel at $10t - 10a$ m/s. Thus his average speed will be $5t$ over that time period, and he'll travel $5t^2$ m in that time. However, Randall is running away at 15m/s , so he'll travel $15t$ feet while the velociraptor travels $5t^2$ feet, and he had a lead of 50m. Thus the distance between them is equal to $5t^2 - 15t - 50$.

This equation may be factored into $5(t-5)(t+2)$, which reaches 0 with t positive at $t = 5$. □

Problem 8. In Moczish, a word is made up of 8 binary digits. A word may be transformed by considering any set of digits in a word such that their sum is even and reversing the order of those digits. For instance, in $00\underline{11}0000$, we can apply this transform to the underlined digits and get 00001100 . Words after transformation have the same meaning as before transformation. What is the greatest number of distinct meanings that can be conveyed in Moczish?

Solution. This problem was based on one from the Leningrad Math Olympiad: 1989, Grade 8, Problem 7. I thought it was pretty cool. This solution is heavily based on the one in Leningrad Mathematical Olympiads 1987-1991.

Consider the 36 words that look like

$$1_a 0_b 10_{7-a-b}$$

for some $a, b \geq 0$ (where d_r represents the digit d repeated r times) and the word 00000000 . We will show that any word w can be transformed into one of these 37 words.

If w has 1 or fewer digits 1, the word is in the set of 37 already. Otherwise, take the substring from the beginning of the word to the second 1 in the word and flip it. This is allowed since the sum is 2. Now if there are more than one 1 in the last 7 digits of the number, flip from the second digit to the second 1 in these last 7 digits, again putting 1s at the beginning. In the end, we will get a number that looks like $1_a 0_b 10_{7-a-b}$.

Now we must prove that the two words $1_a 0_b 10_{7-a-b}$ and $1_c 0_d 10_{7-c-d}$ have the same meaning only if $a = c$ and $b = d$. The operation does not change the number of 1s in the word, thus $a = c$. Further, observe that the number of 0s for which the sum of the digits to the right of the 0 is odd is unchanged by the transformation. But this means that, for the two words to have the same meaning, $b = d$. Thus there are exactly $\boxed{37}$ meanings that can be conveyed in Moczish. \square