Soli Deo Gloria Fall Tournament
Individual Round

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September 22, 2007

Solution. □

1 About the test:

This is a 15-question, 60-minute exam. Pencils of all colors, pens, protractors, rulers, compasses, scissors, tape, watches, and paper of all sorts are allowed. Genii, calculators, PDAs, calculator watches, rocks, pre-inscribed paper, and anything which is deemed annoying by your proctor are disallowed. If you choose to use scissors or tape, you must do so neatly and throw the results and side effects away before you leave.

2 Scoring:

This round is worth a total of 200 points. The problems are not equally valuable; correct answers are worth from 4 to 25 points. Point values are indicated beside the question number. A wrong or blank answer is worth 0 points.

Solution.

1	B
2	D
3	D
4	B
5	D
6	C
7	A
8	D
9	A
10	C
11	A
12	E
13	E
14	D
15	A

□

(4 points) Question 1. Karthik was playing football with his friends. He caught the ball an average of 2.25 times per 15 minutes that he played. If the football game went on for 40 minutes, how many times did he catch the ball?

A) 3 B) 6 C) 9 D) 10 E) 20

Solution. Since Karth catches the ball 2.25 times in 15 minutes, he catches the ball .75 times in 5 minutes, or $8 \cdot .75 = 6$ times in 40 minutes, choice $\boxed{\text{B}}$
. ☐

(6 points) Question 2. Take three mutually tangent circles of radius 1. A regular hexagon is placed such that the centers of the circles are all vertices of the hexagon. Find the area of the region inside the hexagon but outside the circles.

A) 0 B) $\sqrt{3} - \frac{\pi}{2}$ C) $2\sqrt{3} - \frac{\pi}{3}$ D) $2\sqrt{3} - \pi$ E) $3 - \frac{3\pi}{2}$

Solution. Observe that each internal angle of the hexagon has measure of 120 degrees. Then one-third of each circle lies inside the hexagon, so one entire circle lies inside the hexagon. Thus the desired area is the area of the hexagon minus π. We can use 30-60-90 triangles to determine that the hexagon's side length is $\frac{2}{\sqrt{3}}$, so the area of the hexagon is $6\frac{1}{\sqrt{3}} = 2\sqrt{3}$. The area inside the hexagon but outside the circles is then $2\sqrt{3} - \pi$, choice $\boxed{\text{D}}$
. ☐

(7 points) Question 3. Let s be a sequence of numbers such that $s_0 = s_1 = s_2 = 1$ and, for $n \geq 2$, $s_{n+1} = s_n s_{n-1} - s_{n-2}$. Find

$$\sum_{i=0}^{100} s_i$$

A) 4 B) 8 C) 10 D) 11 E) 75

Solution. Note that, in a sequence of numbers such that each subsequent member is only dependent on the previous n members, after n members which have occurred previously repeat the whole sequence will repeat. For instance, in the sequence for which $s_0 = s_1 = 1$ and $s_n = s_{n-1} - s_{n-2}$, we have 1, 1, 0, -1, -1, 0, 1, 1, ... and now the sequence will repeat because 2 members have repeated. This principle is actually used in ZIP compression.

Our sequence goes 1,1,1,0,-1,-1,1,0,1,-1,-1,0,1,1,1,... Thus each 12 elements the sequence repeats. The sum of each 12 is 1+1+1-1-1+1+1-1-1=1. In the first 101 terms, there are 8 sets summing to 2 each, and five more elements: the first five elements are repeated an extra time. Thus the total sum is $8(1) + 1 + 1 + 1 + 0 - 1 = 11$, choice $\boxed{\text{C}}$. $\square$

(10 points) Question 4. There are two distinct right triangles with sides x, $x + 3$, and 20, in some order, for some real numbers x. Find the sum of the lengths of the hypotenuses of these two triangles.
A) $\frac{-3+\sqrt{791}}{2}$ B) $\frac{511}{6}$ C) $\frac{37+\sqrt{791}}{2}$ D) 23 E) $\frac{631}{6}$

Solution. The sides of a triangle are all positive, so $x < x + 3$. Thus we have either $x + 3$ or 20 as the hypotenuse. If $x + 3$ is the hypotenuse, we have $(x)^2 + 20^2 = (x + 3)^2 \to 20^2 = 6x + 9 \to x = \dfrac{391}{6}$. The other possibility is that the hypotenuse has length 20, so the sum of the two lengths is $\dfrac{511}{6}$, choice $\boxed{\text{B}}$. $\square$

(10 points) Question 5. Chuck was working on collecting the complete set of Monty Python episodes when he came across a store sending nothing but Monty Python and Star Wars movies. Unfortunately, this store is magical, much like the Holy Grail, and can only be seen once, and Chuck forgot his credit card, so he can only pay cash. All Monty Python episodes cost the same amount, and all Star Wars movies also cost the same amount. He worked out he could buy 5 Monty Python episodes and 3 Star Wars movies and have 40% of his money left. If he then proceeded to buy 5 more Monty Python episodes he'd have no money left, while if he bought 3 more Star Wars movies, he'd have 5 dollars left. How much money did he start with?
A) 5 B) 6.25 C) 15 D) 25 E) 42

Solution. We know that 5 Monty Python episodes is 40% of Chuck's money, and 5 MP episodes and 3 SW movies is 60% of his money, so 3SW movies is 20% of his money. Since 3SW movies plus 5 dollars is 40% of Chuck's money, \$5 is 20% of his money. Thus Chuck started with 25 dollars, choice $\boxed{\text{D}}$. $\square$

(14 points) Question 6. A pair of integers fulfills all 3 of the following conditions:

1. The sum of the two numbers ends in 3
2. The difference of the two numbers is prime
3. The product of the two numbers is square.

One of the integers in the pair is 9. What is the other?
A) 16 B) 64 C) 4 D) 14 E) 243

Solution. This problem is based on one I saw in the Pre-Olympiad forum on artofproblemsolving.com recently, but I didn't write down the source and I can't find it again. I believe it was a European problem.

Trial and error on the choices gives choice $\boxed{\text{C}}$: 4+9=13, whose units digit is 3; 9-4=5, which is prime; $9 \cdot 4 = 36$ which is square. In fact, (4,9) is the only integral pair that works. □

(8 points) Question 7. In a math tournament, Miles, Linda, Phillip, Ruby, Jordan, William, Carol, Wesley, Mark, and Brayden were the top 10 students, in some order. Mrs. Poss randomly distributed the 10 trophies to them. The probability that exactly 8 people received the correct trophy may be represented as $\frac{a}{10!}$. Find a.
A) $\binom{10}{2}$ B) $8!\binom{10}{2}$ C) $2 \cdot 2\binom{10}{8}$ D) $2 \cdot 8!\binom{10}{2}$ E) $2\binom{10}{8}$

Solution. There are 10! possible arrangements possible disregarding arrangements. For exactly 2 students not to receive their trophies, we choose 8 students to receive their correct trophy in $\binom{10}{8}$ ways, and the remaining two students must have their trophies swapped. Thus the probability is $\frac{\binom{10}{8}}{10!}$, and since $\binom{10}{2} = \binom{10}{8}$, the answer is choice $\boxed{\text{A}}$. □

(12 points) Question 8. The game *H3ar+5* is a 4-person card game. In each round of *H3ar+5* , one player gets at least 13 points, and a total of 26 points are given in each round. For instance, Billy could get 15 points, Danny 6, Tommy 5, and Johnny 0 in one round. The game of *H3ar+5* ends when one player's total score is more than 100 at the end of a round. What is the maximum number of rounds a game of *H3ar+5* could go on?
A) 4 B) 7 C) 15 D) 16 E) 22

Solution. By the Pidgeonhole Principle, if there are 401 or more points distributed among 4 players, one player must have more than 100 points. In 15 rounds, there will be 390 points between the 4 players, so the game might not be over after 15 rounds if the scores are, say, 97-97-98-98. After 16 rounds, however, there will be 416 points between the 4 players, so one player must have more than 100 and the game must end. The answer is thus choice $\boxed{\text{D}}$
. □

(12 points) Question 9. Given that the roots of the equation $x^2+px+3=0$ are positive and not imaginary, find the maximum value of p.
A) $-2\sqrt{3}$ B) $-\sqrt{3}$ C) $\sqrt{3}$ D) $2\sqrt{3}$ E) $\frac{9}{4}$

Solution. Let the roots be a and b. Then we have $3=ab$ and $-p=a+b$. We have a and b both positive, so using AM-GM, we have $\frac{a+b}{2}\geq\sqrt{ab}\rightarrow a+b\geq 2\sqrt{3}$. For the maximum p, we want the minimum value of $a+b$, so the maximum p-value is $-2\sqrt{3}$, choice $\boxed{\text{A}}$. □

(13 points) Question 10. Consider a set of line segments, one each of length 1,2,3,...,9. These may be placed such that the nonagon formed has a circumcircle. How many noncongruent such nonagons are possible?
A) 9! B) 8! C) $\frac{8!}{2}$ D) $\frac{8!}{4}$ E) 7!

Solution. Observe that, for a given radius of the circumcircle, no matter where on the circle a line segment is it will cut off the same arc length. Thus we are simply determining how many arrangements of 1-9 there are around a circular table, where rotations and flips are not counted extra.

There are 9! arrangements, ignoring rotations and flips. However, each unique one is counted 9 times, since we can rotate the circle's arrangement by 9 ways, so we're down to 8! arrangements. Finally, given the position of 1, the circle can be flipped over, and we see that we count each flip twice, so there are 8!/2 noncongruent nonagons, choice $\boxed{\text{C}}$. □

(17 points) Question 11. Michael Dirr wants to fill the 'family tree' shown with the integers 1 through 7, inclusive, in such a way that no number has a child greater than it. For instance, if 3 appears in the topmost circle, 4,5,6, and 7 cannot appear as its children. In how many ways can he fill the tree in this way?
A) 45 B) 90 C) 120 D) 720 E) 5040

Solution. For any tree, the number of ways we can fill the n nodes is simply $n!$ divided by the products of the number of subtrees of each of the nodes. [Consider any of the $n!$ permutations. Obviously, there is a $\frac{1}{n}$ probability that the topmost node is largest. Similarly, we can take any one subtree with q subtrees and there is a $\frac{1}{q}$ probability that the topmost node is largest, as needed.]

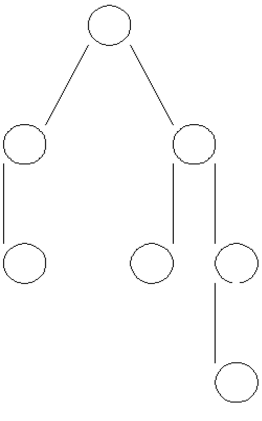

Figure 1: Problem 11: Dr. Dirr's tree.

The lowest node has 1 subtree, itself. In the second row from the bottom, there are 1, 1, and 2 subtrees, from left to right. In the second from the top, there are 2 and 4, respectively, and in the top there are 7. Thus there are $\frac{7!}{1\cdot1\cdot1\cdot1\cdot2\cdot2\cdot4\cdot7} = \boxed{45}$ possibilities, choice $\boxed{\text{A}}$. ☐

(16 points) Question 12. Let $A = (0,0)$,$B = (0,6)$, and $C = (0,8)$. There are two points X where $AX + BX = 10$ and $BX + CX = 10$. What is the y-coordinate of these two points?

A) 2.4 B) 8 C) 6 D) 10 E) 4

Solution. We know that $AX = CX$ by subtracting the two given equations, thus X lies equidistant from A and C. Since $A = (0,0)$ and $C = (0,8)$, $X = (a,4)$ for some a, giving a y-coordinate of 4, choice $\boxed{\text{E}}$. ☐

(20 points) Question 13. A quadrilateral $ABCD$ has $AB = 12$, $BC = 16$, $CD = 10$, $AD = 10\sqrt{3}$, and can be inscribed in a circle with center O. Find the area of $\triangle ABO + \triangle CDO$.

A) 44 B) $24 + \frac{25}{2}\sqrt{3}$ C) 42 D) $44 + \frac{55}{2}\sqrt{3}$ E) $48 + 25\sqrt{3}$

Solution. Consider the two triangles ACD and ABC, which share side AC. Since $ABCD$ is cyclic, $\angle ADC + \angle ABC = 180$, which is fulfilled when they are both equal to 90 degrees. Thus this quadrilateral is simply two right triangles sharing a hypotenuse.

Since O is equidistant from A and C, and ABC is right, O bisects the line AC. Thus $\triangle ABO = \frac{1}{2}\triangle ABC$ and $\triangle CDO = \frac{1}{2}\triangle ACD$, so the sum is equal to half the area of $ABCD$. $[ABC] = \frac{12\cdot16}{2} = 96$, and $[ACD] = \frac{10\cdot10\sqrt{3}}{2} = 50\sqrt{3}$, so the sum is equal to $48 + 25\sqrt{3}$, choice $\boxed{\text{E}}$. ☐

(23 points) Question 14. An equilateral triangle ABC is inscribed in a circle of radius $\frac{10}{\sqrt{3}}$. Points D and E are on sides AB and AC such that AD/AB=1/2 and AE/AC=4/5. BE and CD intersect at point P. AP is extended to meet BC at F and the circle at G. The length of AG may be represented as $\frac{a}{\sqrt{b}}$, where $\gcd(a,b) = 1$. Find $a + b$.

A) 29 B) 31 C) 43 D) 71 E) 111

Solution. Problem thanks to Adeel Khan.

Ceva's Theorem states that $\frac{CE}{AE} \cdot \frac{AD}{DB} \cdot \frac{BF}{FC} = 1$. By substituting in those values which we know, we find $BF/FC = 4$, so $BF = 8$ and $FC = 2$. Now we can use Stewart's Theorem on ABC to determine AF: $8\cdot2\cdot10 + 10\cdot d^2 = 10^2\cdot8 + 10^2\cdot2 \rightarrow d = 2\sqrt{21}$. By Power of a Point on F we have $AF \cdot FG =$

$BF \cdot CF \to 2\sqrt{21} \cdot FG = 8 \cdot 2 \to FG = \frac{8}{\sqrt{21}}$, so $AG = 50\sqrt{21}/21 = \frac{50}{\sqrt{21}}$, so $a+b = 71$,choice $\boxed{\text{D}}$. ☐

(25 points) Question 15. A possibly-degenerate triangle has sides of length a, b, c and total side length of 1. Find the greatest k such that

$$(a-\frac{b}{2})(a-\frac{c}{2})+(b-\frac{a}{2})(b-\frac{c}{2})+(c-\frac{a}{2})(c-\frac{b}{2}) \geq k$$

for all a, b, c.

A) $\frac{1}{12}$ B) 1 C) $\frac{11}{9}$ D) $\frac{3}{2}$ E) 2

Solution. I(Billy like this problem. It isn't geometry, it's an inequality, and I love inequality. Inequality is the basis of politics.

Multiplying both sides of the inequality by 4, we have

$$(2a-b)(2a-c)+(2b-c)(2b-a)+(2c-a)(2c-b) \geq k \cdot 4$$

Expanding the terms, we have

$$4(a^2+b^2+c^2)-3(ab+ac+bc)$$

. We have $ab+ac+bc \implies abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$. The minimum value of the expression occurs when the positive terms are minimized and the negative terms are maximized, and by AM-GM-HM the maximum value of the negative term is

$$max(abc) \cdot max(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) = \frac{1}{27} \cdot 27 = 1$$

The minimum value of $a^2+b^2+c^2$ is $\frac{1}{3}$, by RMS-AM-GM, which states $\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{a+b+c}{3}$. Thus we have $\frac{4}{3} - 1 \geq 4k \to k = \frac{1}{12}$.

This is the maximum k since we can find a set of $(a, b, c) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ that gives $k = \frac{1}{12}$, fulfilling the equality case, and by AM-GM k cannot be lower than $\frac{1}{12}$. The correct choice is thus choice $\boxed{\text{A}}$. ☐