

Soli Deo Gloria Fall Tournament, Team Round

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Solution.

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1 About the test:

This test consists of 4 multi-part questions whose answers depend on each other, to be solved in 40 minutes. You will have to share your answers in order to solve more than the first part of each question. There is a pattern to the dependency. Some work in discovering this pattern may prove helpful in allowing each member of the team to give their answer directly to the next person who needs it. It is possible, though not recommended, for the members of the team to break up the parts such that each member's parts depend only on each other.

Pencils of all colors, pens, protractors, rulers, compasses, scissors, tape, watches, and paper of all sorts are allowed. Genii, calculators, PDAs, calculator watches, rocks, pre-inscribed paper, and anything which is deemed annoying by your proctor are disallowed. If you choose to use scissors or tape, you must do so neatly and throw the results away before you leave.

The answers may be integers or fractions, and must be written on the correct blank on this cover/answer sheet to receive credit. Please write neatly; if there is doubt about your answer it will be counted wrong. Fractions must be expressed in lowest terms.

2 Scoring:

Each question has 4 parts. The first three parts are worth 10 points each, and the last part is worth 20 points.

This round is worth a total of 200 points for your team. Good luck!

1a |

1b |

1c |

1d |

2a |

2b |

2c |

2d |

3a |

3b |

3c |

3d |

4a |

4b |

4c |

4d |

	1a	$\frac{54}{25}$
	1b	0
	1c	12
	1d	$4(\sqrt{3} + \sqrt{15})$
	2a	3160
	2b	56
	2c	16
	2d	3
<i>Solution.</i>	3a	2
	3b	5032
	3c	$\frac{6+\sqrt{66}}{5}$
	3d	1072, 1053
	4a	$\frac{1}{6}$
	4b	347
	4c	509
	4d	142

□

1 Problem 1

1.1 Part a

A triangle ABC has sides of length 3, 4, and 5, with angle B being right and $AB < AC$. An altitude is drawn from B to AC , and the intersection of this altitude and AC is X . What is the area of ABX ?

Solution. We have $\angle AXB = \angle CXB = 90$, thus $\triangle ABX \sim \triangle ACB$. We know that $AB = 3$ and $AC = 5$, thus the ratio of ABX to ACB is $\frac{3}{5}$, and the ratio of the areas is $(\frac{3}{5})^2 = \frac{9}{25}$. The area of ACB is $\frac{3 \cdot 4}{2} = 6$, thus the area of ABX is $\boxed{\frac{54}{25}}$. □

1.2 Part b

Let $\frac{a}{b}$ be the answer to 4a, with $\gcd(a, b) = 1$, and let $T = a + b$. A right triangle has a hypotenuse of length T and the longer leg of length 5. Find the sum of the possible values of $\cos(2\theta)$, where θ is a non-right angle in the right triangle.

Solution. Let the triangle be ABC , with B a right angle and $AB < AC$. Then $\cos 2\angle BAC = \cos^2 \angle BAC - \sin^2 \angle BAC$ and $\cos 2\angle BCA = \cos^2 \angle BCA - \sin^2 \angle BCA$. Now $\sin \angle BAC = \cos \angle BCA$, thus upon addition of the two values we have $0 + 0 = \boxed{0}$. □

1.3 Part c

Let T be the answer to 3b modulus 7 (i.e. the remainder when divided by 7). Right triangle ABC has right angle B and $AB = T$. Right triangle BCD has right angle C . Quadrilateral $ABCD$ may be inscribed in circle O and has radius 5. What is the area of $\triangle OBC$?

Solution. We have $\angle BAC$ cutting off the same arc as $\angle BDC$, thus $\angle BDC = \angle BAC$. Since $\angle BCD = \angle ABC = 90^\circ$, $ABC \cong DCB$. The intersection of AC and BD is the circumcenter, since $ABCD$ is a rectangle, so $AC = BD = 10$. By the Pythagorean Theorem, BC has length $\sqrt{100 - T^2}$, and as O is the midpoint of AC the height of $\triangle OBC$ is $\frac{T}{2}$. Thus the area of $\triangle OBC$ is $\frac{T\sqrt{100 - T^2}}{4}$. The answer to 3b is 5032, so $T = 6$, so our answer is $\boxed{12}$. □

1.4 Part d

Let T be the answer to 2c. Bryan has made a path from his house. It runs in a straight line for T feet before it intersects the road at a 66° angle. It then turns so that it meets the road at a 6° angle, and is heading away from the road (which has width 0). It continues for T more units. If instead he had cut his road only parallel and perpendicular to the straight road, how far would he have cut parallel to the road? Express in the form $a(\sqrt{b} + \sqrt{c})$.

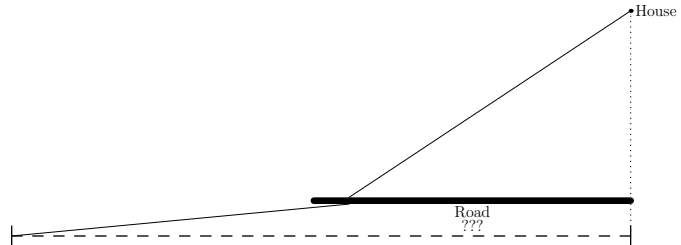


Figure 1: Figure not to scale

Solution. Essentially we are trying to calculate $T(\cos 66 + \cos 6)$. Using the Prosthaphaeresis identity for the sum of two cosines, $\cos \theta + \cos \gamma = 2 \cos \frac{\theta+\gamma}{2} \cos \frac{\theta-\gamma}{2}$, we have the desired sum to be $2T(\cos 30 \cos 36)$. $\cos 30$ is trivially $\frac{\sqrt{3}}{2}$, while $\cos 36 = \frac{\phi}{2} = \frac{1+\sqrt{5}}{4}$, thus the desired sum is $T(\frac{\sqrt{3}+\sqrt{15}}{4})$. The answer to 2c is 16, so the correct answer is $4(\sqrt{3} + \sqrt{15})$. \square

2 Problem 2

2.1 Part a

There are about 80 unique prepositions in the English language. Joe chooses 2 prepositions. How many possible combinations of two words could he choose?

Solution. This is a simple $\binom{80}{2}$ calculation, which is equal to $40 \cdot 79 = \boxed{3160}$. \square

2.2 Part b

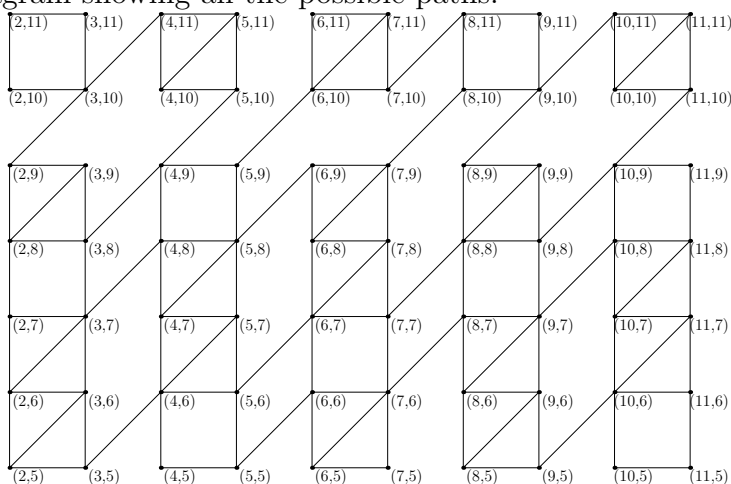
Let the answer to 1a be $\frac{a}{b}$ in reduced form, let $c = a + b$, and let T equal the average of the digits of c . Jack Sprat has T teeth that need to be pulled. How many ways can Dr. Sheen choose 3 teeth in Jack's mouth that need pulling?

Solution. Again, a simple calculation, as long as we get the correct answer for 1a. The answer to 1a is $\frac{54}{25}$, so $c = 79$ and $T = 8$. We then calculate $\binom{8}{3} = \boxed{56}$. \square

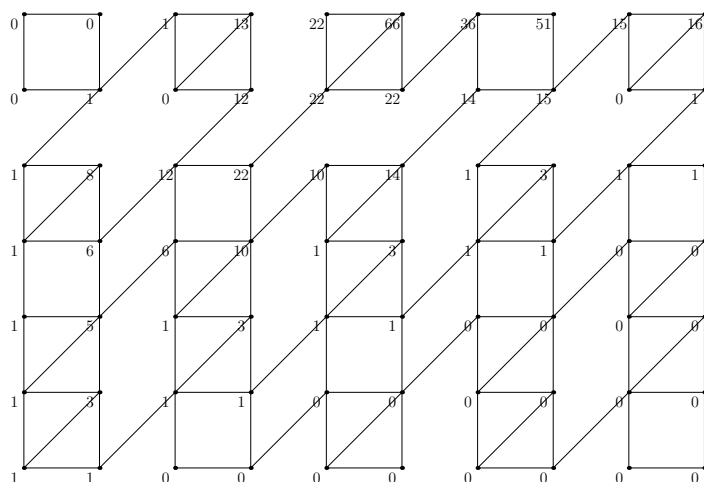
2.3 Part c

A beast is moving about on a Cartesian plane. He begins at the point $(2, T)$, where T is the sum of the digits of the sum of the digits [which is not a typo] of the answer to 4b. If he is on the point (a, b) , he can move to $(a + 1, b + 1)$ iff $3 \nmid (a + b)$, he can move to $(a + 1, b)$ iff $2 \mid a$, or he can move to $(a, b + 1)$ iff $3 \nmid b$ or $2 \mid b$. In how many ways can the beast move to $(11, 11)$?

Solution. This is a lovely problem for casework. We first find $T =$ the sum of the digits of $14 \rightarrow T = 5$. Now we work out the possible moves. We draw a diagram showing all the possible paths:



And then label every node on this diagram with the number of ways to get to it, starting with 1 way to get to $(2,5)$:



Which means there are 16 ways to get to (11,11). □

2.4 Part d

Let $\frac{T+\sqrt{b}}{c}$ be the answer to $3c$, where T, b, c are not necessarily distinct positive integers, b is indivisible by the square of any prime, and $\gcd(b, c) = 1$. The answer should be in the form $\frac{T+\sqrt{b}}{c}$ on your answer sheet already.

John has T dollars. He can buy one apple and two oranges for T dollars. He can buy three apples and one orange for $2T$ dollars. What is the ratio of the price of apples to the price of oranges?

Solution. Let apples cost a and oranges cost b . We then have the equations

$$a + 2b = T$$

$$3a + b = T$$

which give

$$3a + 6b = 3T$$

$$3a + b = 2T$$

which gives $5b = T$, so $b = \frac{T}{5}$. Now going back to the first equation we have $a + \frac{2T}{5} = T$, thus $a = \frac{3T}{5}$ and $a/b = \boxed{3}$. □

3 Problem 3

3.1 Part a

The equation $x^3 - x^2 + 3x - 10 = 0$ has 1 rational root. Find this root.

Solution. We note that the equation factors to $(x - 2)(x^2 + x + 5)$, the only rational root of which is $\boxed{2}$. \square

3.2 Part b

Let T equal the sum of the digits of the answer to 2a. $p(x)$ is a polynomial of degree 3 such that:

$$p(-1) = -6$$

$$p(0) = 2$$

$$p(1) = T$$

$$p(3) = 146$$

Find $p(10)$.

Solution. The answer to 2a is 3160, thus $T = 10$. WOLOG let $p(x) = ax^3 + bx^2 + cx + d$. We then have the system of equations:

$$-a + b - c + d = -6 \tag{1}$$

$$d = 2 \tag{2}$$

$$a + b + c + d = 10 \tag{3}$$

$$27a + 9b + 3c + d = 146 \tag{4}$$

$$\tag{5}$$

Trivially by the second we have $d = 2$. By adding the first and third equations, we have $2(b + d) = 4$, thus $b = 0$. Plugging these numbers back into the first or third equations, we have $a + c = 8$. Finally, plugging in our numbers into the last equation, we have $8a = 40$, so $a = 5$ and so $c = 3$. Thus $p(x) = 5x^3 + 0x^2 + 3x + 2$, and $p(10) = \boxed{5032}$. \square

3.3 Part c

Let T equal the answer to 1b. There is a unique x such that $\log_8 x + \log_4 x + \log_x \frac{1}{2} = T + 2$. Solve for $\log_2 x$.

Solution. Let $\log_2 x = a$, for some a . Then we have

$$\begin{aligned}\frac{a}{3} + \frac{a}{2} + \frac{\log_2 \frac{1}{2}}{a} &= 2 \\ \frac{a}{3} + \frac{a}{2} - \frac{1}{a} &= 2 \\ 5a^2 - 12a - 6 &= 0 \\ a &= \frac{12 \pm \sqrt{144 + 120}}{10}\end{aligned}$$

Since $a > 0$ (else we can't take a logarithm base a) we must have $a =$

$$\boxed{\frac{6 + \sqrt{66}}{5}}$$

□

3.4 Part d

Let T equal the answer to 4c. Find the positive integers for which the sum of the squares of their digits is $2T$ less than that integer.

Solution. This problem is based on USAMTS Year 18 Round 2 Problem 1, with 2006 replaced by $2T$. Solution similar to one of the official solutions by Carlos Dominiguez.

We desire to find the positive integers such that the sum of the squares of their digits is 1018 less than the number. Obviously the solutions are greater than 1018. If they had 5 digits, the maximum value of the sum of the square of their digits is $5 \cdot 9^2 = 405$, so the solutions are less than 10000. If it has 4 digits, the maximum value of the sum of the squares is $4 \cdot 9^2 = 324$, so the solutions are less than 2000 also. Thus the first digit of the solutions must be 1.

Case 1: The second digit is 0. We obtain the system of equations $1 + 0 + c^2 + d^2 + 1018 = 1000 + 10c + d \rightarrow (c - 5)^2 + (d - \frac{1}{2})^2 = \frac{25}{4} \rightarrow (2c - 10)^2 + (2d - 1)^2 = 25$. 25 may be expressed as a sum of two squares in two ways: either $3^2 + 4^2$ or $5^2 + 0^2$. Now the term $(2c - 10)^2$ must be even, thus we have either $(c, d) = (7, 2)$ or $(c, d) = (5, 3)$, corresponding to the numbers $\boxed{1072, 1053}$.

Case 2: The second digit is 1 or greater. The maximum value of the LHS is $c^2 + d^2 + 1100$, and the minimum value of the RHS is $1100 + 10c + d$. This ends up factoring, in the same manner as in case 1, to $(c - 5)^2 + (d - \frac{1}{2})^2 < 0$. But this is impossible, thus the only solutions are the previously-found 1072 and 1053. □

4 Problem 4

4.1 Part a

What is the probability a randomly-selected positive integer is divisible by 8 or 12?

Solution. This is a straightforward problem: in every $\text{lcm}(8, 12)$ consecutive positive integers, there must be $\text{lcm}(8, 12)/8$ positive integers divisible by 8, $\text{lcm}(8, 12)/12$ positive integers divisible by 12, and $\text{lcm}(8, 12)/\text{lcm}(8, 12)$ positive integers divisible by both, so in every 24 consecutive positive integers there are $3+2-1=4$ divisible by 8 or 12. Thus the probability that any randomly-chosen integer is divisible by 8 or 12 is $\frac{4}{24} = \boxed{\frac{1}{6}}$ \square

4.2 Part b

Let T be the answer to 3a. The sum

$$\sum_{i=0}^{16T^2} \sum_{j=0}^i 5^j$$

can be expressed as $\frac{5^b - c}{d}$, where $\text{gcd}(5^b, c) = 5$. Find the sum $b + c + d$.

Solution. We find $T = 2$, so we are to compute the sum

$$\sum_{i=0}^{64} \sum_{j=0}^i 5^j$$

The inner sum is equal to

$$\frac{5^{i+1} - 1}{4}$$

by the formula for a geometric series, so our sum is now

$$\frac{1}{4} \sum_{i=0}^{64} 5^{i+1} - 1 \rightarrow -\frac{65}{4} + \frac{1}{4} \sum_{i=1}^{65} 5^i$$

Applying the same formula, we have

$$-\frac{65}{4} + \frac{5}{16}(5^{65} - 1)$$

which we can massage into the desired form:

$$\frac{-65 \cdot 4 + 5^{66} - 5}{16} = \frac{5^{66} - 265}{16}$$

The desired sum is thus $\boxed{347}$. □

4.3 Part c

Let T be the answer to 2b. Find the second smallest solution in positive integers to $x^2 - Ty^2 = 1$. What is the sum of x and y ? 449,60-;509

Solution. This is a classic Diophantine equation. We find $T = 56$, so our equation is $x^2 - 56y^2 = 1$. Obviously $x \gg y$, so we try to find a solution for y via brute-force. $y = 1$ doesn't work; 57 is not a square. On the other hand, for $y = 2$ $56 \cdot 4 + 1 = 225 = 15^2$, thus the smallest solution is $(15, 2)$.

Now it is easy to prove that, for the special Diophantine equation $x^2 + ay^2 = 1$ with one solution of (b, c) , that $(b^2 + ac^2, 2bc)$ is also a solution. For (b, c) the smallest solution to this Diophantine equation, the other solutions are given by the coefficients of 1 and \sqrt{a} in

$$(b + c\sqrt{a})^m, m = 1, 2, 3, \dots$$

For our equation, the solutions are given by $(15 + 2\sqrt{56})^m$. The second smallest is based on $(15 + 2\sqrt{56})^2 = 449 + 60\sqrt{56}$, giving the second solution to be $(449, 60)$. The desired sum is then $\boxed{509}$. □

4.4 Part d

Let T be the answer to part 1c. For how many positive integers n does $\lfloor \log_n T^2 \rfloor \mid T^2$?

Solution. $T = 12$. The prime factorization of 12^2 is $2^4 3^2$. Now there is no integer for which $\lfloor \log_n T^2 \rfloor > 7$, thus we need concern ourselves only with those cases when $\lfloor \log_n T^2 \rfloor \in \{1, 2, 3, 4, 6\}$.

Case 1: $\lfloor \log_n T^2 \rfloor = 1$ This is true for any n greater than 12, for $144-12=132$ values.

Case 2: $\lfloor \log_n T^2 \rfloor = 2$ The smallest integer whose 3rd power is less than 144 is 5, thus there are $12-5=7$ values.

Case 3: $\lfloor \log_n T^2 \rfloor = 3$ The smallest integer whose 4th power is less than 144 is 3, thus there are $5-3=2$ values.

Case 4: $\lfloor \log_n T^2 \rfloor = 4$ The 4th power of 3 is 81, while the 5th is 243, thus $\lfloor \log_3 144 \rfloor = 4$. 1 more value.

Case 5: $\lfloor \log_n T^2 \rfloor = 6$ The only remaining values are 1 and 2: $\log_1 x$ is undefined, and $\lfloor \log_2 144 \rfloor = 7$, thus neither 2 nor 1 satisfies the desired equation.

Thus in sum only two values of n less than or equal to 144 do not satisfy the divisibility requirement: 2 and 1. There are 142 values which do. \square