Soli Deo Gloria Fall Tournament, Team Round

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1 About the test:

This test consists of 4 multi-part questions whose answers depend on each other, to be solved in 40 minutes. You will have to share your answers in order to solve more than the first part of each question. There is a pattern to the dependency. Some work in discovering this pattern may prove helpful in allowing each member of the team to give their answer directly to the next person who needs it. It is possible, though not recommended, for the members of the team to break up the parts such that each member's parts depend only on each other.

Pencils of all colors, pens, protractors, rulers, compasses, scissors, tape, watches, and paper of all sorts are allowed. Genii, calculators, PDAs, calculator watches, rocks, pre-inscribed paper, and anything which is deemed annoying by your proctor are disallowed. If you choose to use scissors or tape, you must do so neatly and throw the results away before you leave.

The answers may be integers or fractions, and must be written on the correct blank on this cover/answer sheet to receive credit. Please write neatly; if there is doubt about your answer it will be counted wrong. Fractions must be expressed in lowest terms.

2 Scoring:

Each question has 4 parts. The first three parts are worth 10 points each, and the last part is worth 20 points.

This round is worth a total of 200 points for your team. Good luck!

1a	
1b	
1c	
1d	
2a	
2b	
2c	
2d	
3a	
3b	
3c	
3d	
4a	
4b	
4c	
4d	

1 Problem 1

1.1 Part a

A triangle ABC has sides of length 3, 4, and 5, with angle B being right and $AB < AC$. An altitude is drawn from B to AC, and the intersection of this altitude and AC is X. What is the area of ABX?

1.2 Part b

Let $\frac{a}{b}$ be the answer to 4a, with $gcd(a, b) = 1$, and let $T = a + b$. A right triangle has a hypotenuse of length T and the longer leg of length 5. Find the sum of the possible values of $\cos(2\theta)$, where θ is a non-right angle in the right triangle.

1.3 Part c

Let T be the answer to 3b modulus 7 (i.e. the remainder when divided by 7). Right triangle ABC has right angle B and $AB = T$. Right triangle BCD has right angle C. Quadrilateral $ABCD$ may be inscribed in circle O and has radius 5. What is the area of $\triangle OBC$?

1.4 Part d

Let T be the answer to 2c. Bryan has made a path from his house. It runs in a straight line for T feet before it intersects the road at a 66° angle. It then turns so that it meets the road at a 6° angle, and is heading away from the road (which has width 0). It continues for T more units. If instead he had cut his road only parallel and perpindicular to the straight road, how far would he have cut parallel to the road? Express in the form $a(\sqrt{b} + \sqrt{c})$.

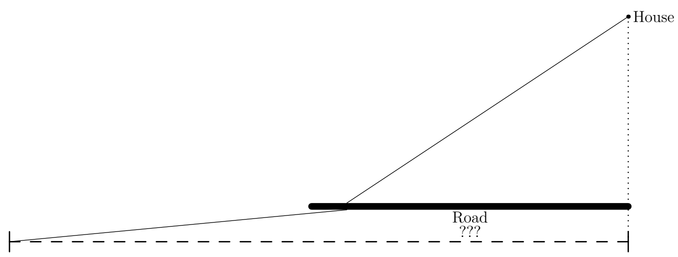

Figure 1: Figure not to scale

2 Problem 2

2.1 Part a

There are about 80 unique prepositions in the English language. Joe chooses 2 prepositions. How many possible combinations of two words could he choose?

2.2 Part b

Let the answer to 1a be $\frac{a}{b}$ in reduced form, let $c = a + b$, and let T equal the average of the digits of c. Jack Sprat has T teeth that need to be pulled. How many ways can Dr. Sheen choose 3 teeth in Jack's mouth that need pulling?

2.3 Part c

A beast is moving about on a Cartesian plane. He begins at the point $(2, T)$, where T is the sum of the digits of the sum of the digits [which is not a typo] of the answer to 4b. If he is on the point (a, b), he can move to $(a+1, b+1)$ iff $3 \nmid (a + b)$, he can move to $(a + 1, b)$ iff $2 \mid a$, or he can move to $(a, b+1)$ iff $3 \nmid b$ or $2 \mid b$. In how many ways can the beast move to $(11, 11)$?

2.4 Part d

Let $\frac{T+\sqrt{b}}{c}$ be the answer to 3c, where T, b, c are not necessarily distinct positive integers, b is indivisible by the square of any prime, and $\gcd(b, c) = 1$. The answer should be in the form $\frac{T+\sqrt{b}}{c}$ on your answer sheet already.

John has T dollars. He can buy one apple and two oranges for T dollars. He can buy three apples and one orange for $2T$ dollars. What is the ratio of the price of apples to the price of oranges?

3 Problem 3

3.1 Part a

The equation $x^3 - x^2 + 3x - 10 = 0$ has 1 rational root. Find this root.

3.2 Part b

Let T equal the sum of the digits of the answer to 2a. $p(x)$ is a polynomial of degree 3 such that:
$p(-1) = -6$
$p(0) = 2$
$p(1) = T$
$p(3) = 146$

Find $p(10)$.

3.3 Part c

Let T equal the answer to 1b. There is a unique x such that $\log_8 x + \log_4 x + \log_x \frac{1}{2} = T + 2$. Solve for $\log_2 x$.

3.4 Part d

Let T equal the answer to 4c. Find the positive integers for which the sum of the squares of their digits is $2T$ less than that integer.

4 Problem 4

4.1 Part a

What is the probability a randomly-selected positive integer is divisible by 8 or 12?

4.2 Part b

Let T be the answer to 3a. The sum

$$\sum_{i=0}^{16T^2} \sum_{j=0}^{i} 5^j$$

can be expressed as $\frac{5^b - c}{d}$, where $gcd(5^b, c) = 5$. Find the sum $b + c + d$.

4.3 Part c

Let T be the answer to 2b. Find the second smallest solution in positive integers to $x^2 - Ty^2 = 1$. What is the sum of x and y? 449,60-¿509

4.4 Part d

Let T be the answer to part 1c. For how many positive integers n does $\lfloor \log_n T^2 \rfloor \mid T^2$?