Dear Problem Solvers,

It has been a pleasure to write much of the 2007 iTest. The creators of the iTest had a great idea, and with better and better execution, their idea has grown into something that is hopefully enjoyable and valuable not only to many contest participants, but also thousands more in the future. I am particularly grateful for the opportunity to make slight alterations to the testing format, which freed me to do a few things with the iTest that are very hard to do with any contest:

- Include several problems that challenge the abilities of problem solvers, regardless of whether they are new to contest problem solving or training for a high score on an olympiad. Every day is a chance to dig deeper into the realm of problems that can be solved!
- Include problems that are harder than the usual fare, but mostly insofar as they require more investigation, not necessarily more complex tools. I believe this aspect of the iTest allows it to reach minds in a way like few other contests. Perhaps none. If it serves as a bridge for some problem solvers to step deeper into mathematics, writing proofs and contests such as the USAMTS or USAMO or participating in real mathematical research, then it will have accomplished something rare and enormously valuable.
- Problems that include little pieces of stories that hopefully engage solvers and encourage everyone to think about how problem solving can be applied to everyday life, even if the examples are sometimes artificial.
- Abandon even norms for most mathematics contests in which areas of mathematics such as algebra, counting methods, geometry, number theory, and probability make up the bulk of the problems (and perhaps reasonably so!). I believe that the ability to scatter other ideas into "mini-research" problems provides a unique educational opportunity for many participants.

In trying to achieve these goals, I selected a few old problems (sometimes inserting my own twists), wrote a few that fit classic problem styles, and dove into a few topics that just seemed interesting – and found them very rewarding! I hope as many people as possible share the joys of these insights.

It should be noted that Princeton University undergraduate Nathan Savir wrote about half of the problems and solutions for the multiple choice portion of the 2007 iTest, and Harvard University undergraduate Zachary Abel wrote the three hardest geometry problems in the rest of the exam, along with their solutions, and created other diagrams that make this solution guide more illuminating. Their efforts have been instrumental in putting together a diverse test of creativity, ingenuity, and determination. We hope both that you enjoyed the exam and that you learn something new from the following solution manual.

– Mathew Crawford

1. A twin prime pair is a pair of primes (p, q) such that $q = p + 2$. The Twin Prime Conjecture states that there are infinitely many twin prime pairs. What is the arithmetic mean of the two primes in the smallest twin prime pair? (1 is not a prime.)

   (A) 4

   Answer: (A) 4

   Solution: The smallest twin prime pair is (3, 5). The arithmetic mean of 3 and 5 is 4.

2. Find the value of $a + b$ given that (a, b) is a solution to the system

$$3a + 7b = 1977,$$
$$5a + b = 2007.$$

   (A) 488 (B) 498

   Answer: (B) 498

   Solution 1: We can solve the system of equations, then plug the result into the expression we want to evaluate. Either elimination or substitution would work, but we choose substitution as the second equation in the system looks like a nice substitution equation, if we manipulate it just a bit:

$$b = 2007 - 5a.$$

   Plugging in for b in the first equation, we get

$$3a + 7(2007 - 5a) = 1977 \qquad \Leftrightarrow \qquad -32a + 14049 = 1977.$$

   So, $32a = 12072$, and we get $a = 377.25$. Thus also,

$$b = 2007 - 5a = 2007 - 5(377.25) = 2007 - 1886.25 = 120.75.$$

   Finally, $a + b = 377.25 + 120.75 = 498$.

   Solution 2: Adding the equations yields $8a + 8b = 3984$. Dividing this new equation by 8, we get $a + b = 498$.

3. An *abundant number* is a natural number, the sum of whose proper divisors is greater than the number itself. For instance, 12 is an abundant number:

$$1 + 2 + 3 + 4 + 6 = 16 > 12.$$

However, 8 is not an abundant number:

$$1 + 2 + 4 = 7 < 8.$$

Which one of the following natural numbers is an abundant number?

(A) 14 (B) 28 (C) 56

Answer: (C) 56

Solution 1: We find and add together the proper divisors of each integer:

$$\begin{aligned} 1 + 2 + 7 = 10 &< 14, \\ 1 + 2 + 4 + 7 + 14 = 28 &= 28, \\ 1 + 2 + 4 + 7 + 8 + 14 + 28 = 64 &> 56. \end{aligned}$$

As we see, the sum of the proper divisors of 14 is less than 14, so 14 is not abundant. We call number such as 14 *deficient numbers*. The sum of the proper divisors of 28 is equal to 28. We call such numbers *perfect numbers*. The sum of the proper divisors of 56 is greater than 56, so 56 is an abundant number.

Solution 2: Some students might notice a relationship between the divisors of the given related integers:

$$1 + 2 + 4 + 7 + 14 = (1 + 7) + (2 + 4 + 14) = (1 + 7) + 2(1 + 2 + 7) = 8 + 2(10) = 28.$$

Here, we view the proper divisors of 28 in terms of all the proper divisors of 14. Since $28 = 2 \cdot 14$, each proper divisor of 14 corresponds to a proper divisor of 28 which is twice as large. Adding in the odd proper divisors of 28, we see that the sum of the proper divisors of 28 is more than twice the sum of the proper divisors of 14. Likewise, the sum of the proper divisors of 56 is more than twice the sum of the proper divisors of 28. So, if only one of the three integers is abundant, it must be 56.

How far can you generalize this relationship?

4. Star flips a quarter four times. Find the probability that the quarter lands heads exactly twice.

(A) $\frac{1}{8}$ (B) $\frac{3}{16}$ (C) $\frac{3}{8}$

(D) $\frac{1}{2}$

Answer: (C) $\frac{3}{8}$

Solution: In order to find the probability that Star flipped 2 heads and 2 tails, we count both the number of ways in which this result could occur, and also the total number of sequences of outcomes.

The number of ways to flip 2 heads from 4 flips is

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = 6.$$

The total number of sequences of 4 flips is

$$2^4 = 16.$$

The probability Star flips 2 heads and 2 tails is therefore

$$\frac{6}{16} = \frac{3}{8}.$$

5. Compute the sum of all twenty-one terms of the geometric series

$$1 + 2 + 4 + 8 + \cdots + 1048576.$$

(A) 2097149 (B) 2097151 (C) 2097153

(D) 2097157 (E) 2097161

Answer: (B) 2097151

Solution: The common ratio of the series is 2. We could rewrite the series as

$$2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{20}.$$

We could at this point simply add together all the powers of 2, but there is a nicer way to sum a geometric series. First, we give the sum a name:

$$S = 2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{20}. \tag{1}$$

Now, we multiply this new equation by 2, which gives us the sum of a very similar geometric series:

$$2S = 2^1 + 2^2 + 2^3 + 2^4 + \cdots + 2^{21}. \tag{2}$$

Now, we use the similarity between these two series to get rid of most of the terms we'd need to compute. Subtracting 1 from 2, we get

$$\begin{aligned} S &= 2S - S = (2^1 + 2^2 + \cdots + 2^{21}) - (2^0 + 2^1 + \cdots + 2^{20}) \\ &= (2^1 + 2^2 + \cdots + 2^{20}) + 2^{21} - 2^0 - (2^1 + 2^2 + \cdots + 2^{20}) \\ &= 2^{21} - 2^0 + (2^1 + 2^2 + \cdots + 2^{20}) - (2^1 + 2^2 + \cdots + 2^{20}) \\ &= 2^{21} - 2^0 = 2^{21} - 1. \end{aligned}$$

Now, we compute 2^{21} using the value of 2^{20} that is computed for us in the problem:

$$2^{21} - 1 = 2(2^{20}) - 1 = 2(1048576) - 1 = 2097152 - 1 = 2097151.$$

6. Find the units digit of the sum

$$(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2 + \cdots + (2007!)^2.$$

(A) 0 (B) 1 (C) 3

(D) 5 (E) 7 (F) 9

Answer: (E) 7

Solution: The units digit of any multiple of 10 is 0. After the first four terms of the given series, all the terms have units digit 0. Consider,

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

is a multiple of $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ for all $n \geq 5$. So, $n!$ is a multiple of 10, and therefore, so is $(n!)^2$.

The units digit of the sum of a group of integers is the sum of those integers' units digits. So, first, we compute

$$\begin{aligned} (1!)^2 &= 1^2 = 1, \\ (2!)^2 &= 2^2 = 4, \\ (3!)^2 &= 6^2 = 36, \\ (4!)^2 &= 24^2 = 576. \end{aligned}$$

The rest of the terms have units digit 0, so we are looking for the units digit of

$$1 + 4 + 36 + 576 + 0 + 0 + \cdots + 0 + 0 = 617,$$

which is 7.

7. An equilateral triangle with side length 1 has the same area as a square with side length s. Find s.

(A) $\frac{\sqrt[4]{3}}{2}$ (B) $\frac{\sqrt[4]{3}}{\sqrt{2}}$ (C) 1

(D) $\frac{3}{4}$ (E) $\frac{4}{3}$ (F) $\sqrt{3}$

(G) $\frac{\sqrt{6}}{2}$

Answer: (A) $\frac{\sqrt[4]{3}}{2}$

Solution: The area of an equilateral triangle with side length t is $\frac{\sqrt{3}t^2}{4}$. In this case $t = 1$, and we are solving for

$$s^2 = \frac{\sqrt{3}}{4} \qquad \Rightarrow \qquad s = \frac{\sqrt[4]{3}}{2}.$$

8. Joe is right at the middle of a train tunnel and he realizes that a train is coming. The train travels at a speed of 50 miles per hour, and Joe can run at a speed of 10 miles per hour. Joe hears the train whistle when the train is a half mile from the point where it will enter the tunnel. At that point in time, Joe can run toward the train and just exit the tunnel as the train meets him. Instead, Joe runs away from the train when he hears the whistle. How many seconds does he have to spare (before the train is upon him) when he gets to the tunnel entrance?

(A) 7.2	(B) 14.4	(C) 36
(D) 10	(E) 12	(F) 2.4
(G) 25.2	(H) 123456789	

Answer: (B) 14.4

Solution: The train will reach the front of the tunnel in 1/100 hour, so the tunnel is 1/5 mile long. The train will reach the entrance of the tunnel when Joe reaches the other end, so Joe has 1/250 hours $= 72/5 = 14.4$ seconds.

9. Suppose that m and n are positive integers such that $m < n$, the geometric mean of m and n is greater than 2007, and the arithmetic mean of m and n is less than 2007. How many pairs (m, n) satisfy these conditions?

(A) 0	(B) 1	(C) 2
(D) 3	(E) 4	(F) 5
(G) 6	(H) 7	(I) 2007

Answer: (A) 0

Solution: Unfortunately, the arithmetic mean of two positive numbers is always greater than or equal to the geometric mean, so no such pairs of integers exist.

10. My grandparents are Arthur, Bertha, Christoph, and Dolores. My oldest grandparent is only 4 years older than my youngest grandparent. Each grandfather is two years older than his wife. If Bertha is younger than Dolores, what is the difference between Bertha's age and the mean of my grandparents' ages?

(A) 0	(B) 1	(C) 2
(D) 3	(E) 4	(F) 5
(G) 6	(H) 7	(I) 8
(J) 2007		

Answer: (C) 2

Solution: Since each grandfather is older than his wife, the oldest grandparent is a grandfather and the youngest is a grandmother. Dolores must be married to the older grandfather. Call his age n. Then Dolores' age is $n - 2$. Bertha must be the youngest, so we know her age is $n - 4$. Then her husband's age is also $n - 2$. The mean of the four ages is $n - 2$, so the difference is 2.

11. Consider the "tower of power" $2^{2^{\cdot^{\cdot^{2}}}}$, where there are 2007 twos including the base. What is the last (units) digit of this number?

(A) 0	(B) 1	(C) 2
(D) 3	(E) 4	(F) 5
(G) 6	(H) 7	(I) 8
(J) 9	(K) 2007	

Answer: (G) 6

Solution: Note that the last digit of powers of 2 repeats every fourth term (2, 4, 8, 6, 2, 4, 8, 6, ...). The number to which we raise 2 here is obviously a multiple of 4, so the last digit of the expression will be 6.

12. My frisbee group often calls "best of five" to finish our games when it's getting dark, since we don't keep score. The game ends after one of the two teams scores three points (total, not necessarily consecutive). If every possible sequence of scores is equally likely, what is the expected score of the losing team?

(A) 2/3	(B) 1	(C) 3/2
(D) 8/5	(E) 5/8	(F) 2
(G) 0	(H) 5/2	(I) 2/5
(J) 3/4	(K) 4/3	(L) 2007

Answer: (C) 3/2

Solution: There are three possible final scores: 3-0, 3-1, and 3-2. Call the teams A and B and assume that team A wins (by symmetry, this is all we have to do). We will count the number of ways that each outcome can happen. 3-0 can happen in only one way. 3-1 can happen in 3 ways (the last point has to be by team A). 3-2 can happen in $4!/(2!2!) = 6$ ways. We are given that all of these have equal probability, so the expected score of the losing team is

$$\frac{6(2) + 3(1) + 1(0)}{10} = \frac{15}{10} = \frac{3}{2}.$$

13. What is the smallest positive integer k such that the number $\binom{2k}{k}$ ends in two zeros?

(A) 3 (B) 4 (C) 5

(D) 6 (E) 7 (F) 8

(G) 9 (H) 10 (I) 11

(J) 12 (K) 13 (L) 14

(M) 2007

Answer: (K) 13

Solution: We want 100 to divide $\binom{2k}{k}$. Let's count the number of multiples of 5 in the numerator and the denominator. The numerator has $\lfloor \frac{2k}{5} \rfloor + \lfloor \frac{2k}{25} \rfloor + \cdots$ and the denominator has $2\lfloor \frac{k}{5} \rfloor + 2\lfloor \frac{k}{25} \rfloor + \cdots$, where $\lfloor x \rfloor$ represents the largest integer smaller than x (i.e. we round down). We want the first expression to be larger than the second by 2. Clearly $\lfloor \frac{2k}{5^n} \rfloor$ is no more than one larger than $2\lfloor \frac{k}{5^n} \rfloor$, so to get 2 we need $2k$ to be divisible by 25 one more time than k is. The first time that happens is for $k = 13$. Using the same idea, we easily check that the power of 2 dividing the numerator is more than two larger than the power of 2 dividing the denominator, so the answer is 13.

14. Let $\phi(n)$ be the number of positive integers $k < n$ which are relatively prime to n. For how many distinct values of n is $\phi(n)$ equal to 12?

(A) 0 (B) 1 (C) 2

(D) 3 (E) 4 (F) 5

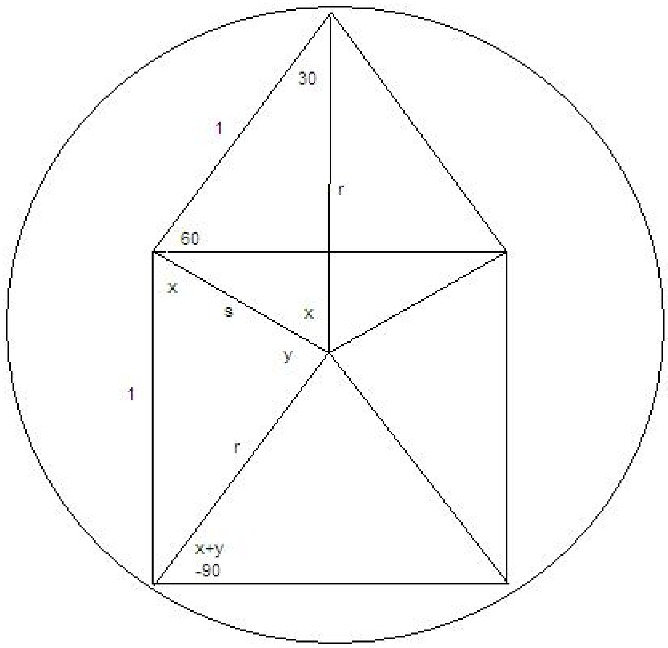
(G) 6 (H) 7 (I) 8

(J) 9 (K) 10 (L) 11

(M) 12 (N) 13

Answer: (G) 6

Solution: If $p_1, p_2, \ldots, p_k$ are the distinct prime factors of n, then $\phi(n) = n(1 - 1/p_1)(1 - 1/p_2)\cdots(1 - 1/p_k)$. Suppose that this equals 12. If n has no repeated prime factors, then $(p_1 - 1)(p_2 - 1)\cdots(p_k - 1) = 12$. Check all possible factorizations of 12: $12, 1 \cdot 12, 2 \cdot 6, 1 \cdot 2 \cdot 6, 3 \cdot 4, 1 \cdot 3 \cdot 4, 2 \cdot 2 \cdot 3, 1 \cdot 2 \cdot 2 \cdot 3$. This gives $n = 13, 21, 26, 42$. If n has one repeated prime factor (repeated once), then $p_1(p_1 - 1)(p_2 - 1)\cdots(p_k - 1) = 12$. This can be achieved with $n = 28$. If n has two repeated prime factors, then $p_1(p_1 - 1)p_2(p_2 - 1) = 12$. This can be satisfied by $n = 36$. It is easy to check that 12 doesn't have enough factors for any other possibility to work. So the answer is 6.



15. Form a pentagon by taking a square of side length 1 and an equilateral triangle of side length 1 and placing the triangle so that one of its sides coincides with a side of the square. Then "circumscribe" a circle around the pentagon, passing through three of its vertices, so that the circle passes through exactly one vertex of the equilateral triangle, and exactly two vertices of the square. What is the radius of the circle?

(A) $\frac{2}{3}$	(B) $\frac{3}{4}$	(C) 1
(D) $\frac{5}{4}$	(E) $\frac{4}{3}$	(F) $\frac{\sqrt{2}}{2}$
(G) $\frac{\sqrt{3}}{2}$	(H) $\sqrt{2}$	(I) $\sqrt{3}$
(J) $\frac{1+\sqrt{3}}{2}$	(K) $\frac{2+\sqrt{6}}{2}$	(L) $\frac{7}{6}$
(M) $\frac{2+\sqrt{6}}{4}$	(N) $\frac{4}{5}$	(O) 2007

Answer: (C) 1

Solution: We see that the two triangles with side lengths labeled are congruent, so $x = y$ and $x = 150 - x$. Then the triangle with vertex 30 is isosceles and $r = 1$.

16. How many lattice points lie within or on the border of the circle defined in the xy-plane by the equation $x^2 + y^2 = 100$?

(A) 1	(B) 2	(C) 4
(D) 5	(E) 41	(F) 42
(G) 69	(H) 76	(I) 130
(J) 133	(K) 233	(L) 311
(M) 317	(N) 420	(O) 520
(P) 2007		

Answer: (M) 317

Solution: We use symmetry to simplify the problem a bit. First, we note that the origin and the 40 points $(\pm n, 0)$ and $(0, \pm n)$ are on or within the circle for integers n such that $1 \leq n \leq 10$. That's 41 lattice points in or on the circle already. Now, we divide the remaining lattice points into regions by quadrant. Since the total number in each quadrant is the same, we just count those in Quadrant I and multiply that total by 4.

Casework helps us as we count the possibilities. We are looking for positive integers x and y such that $x^2 + y^2 \leq 100$, so we consider the cases for x:

$$\begin{array}{lllll} x = 1 & \Rightarrow & y^2 \leq 99 & \Rightarrow & y \leq 9, \\ x = 2 & \Rightarrow & y^2 \leq 96 & \Rightarrow & y \leq 9, \\ x = 3 & \Rightarrow & y^2 \leq 91 & \Rightarrow & y \leq 9, \\ x = 4 & \Rightarrow & y^2 \leq 84 & \Rightarrow & y \leq 9, \\ x = 5 & \Rightarrow & y^2 \leq 75 & \Rightarrow & y \leq 8, \\ x = 6 & \Rightarrow & y^2 \leq 64 & \Rightarrow & y \leq 8, \\ x = 7 & \Rightarrow & y^2 \leq 51 & \Rightarrow & y \leq 7, \\ x = 8 & \Rightarrow & y^2 \leq 36 & \Rightarrow & y \leq 6, \\ x = 9 & \Rightarrow & y^2 \leq 19 & \Rightarrow & y \leq 4. \end{array}$$

So, there are 9 lattice points $(1, y)$, another 9 of the form $(2, y)$, etc., for a total of

$$9 + 9 + 9 + 9 + 8 + 8 + 7 + 6 + 4 = 69.$$

We multiply this total by 4 to get 276, and add the 41 lattice points from the axes to get $276 + 41 = 317$.

17. If x and y are acute angles such that $x + y = \pi/4$ and $\tan y = 1/6$, find the value of $\tan x$.

(A) $\frac{37\sqrt{2} - 18}{71}$ (B) $\frac{35\sqrt{2} - 6}{71}$ (C) $\frac{35\sqrt{3} + 12}{33}$

(D) $\frac{37\sqrt{3} + 24}{33}$ (E) 1 (F) $\frac{5}{7}$

(G) $\frac{3}{7}$ (H) 6 (I) $\frac{1}{6}$

(J) $\frac{1}{2}$ (K) $\frac{6}{7}$ (L) $\frac{4}{7}$

(M) $\sqrt{3}$ (N) $\frac{\sqrt{3}}{3}$ (O) $\frac{5}{6}$

(P) $\frac{2}{3}$ (Q) $\frac{1}{2007}$

Answer: (F) $\frac{5}{7}$

Solution: We know the tangent of y, and we can easily see that the tangent of $x + y$ is 1. So, we can make use of the tangent addition formula:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Plugging in the values we know, we have

$$1 = \frac{\tan x + \frac{1}{6}}{1 - \frac{1}{6}\tan x} \quad \Leftrightarrow \quad 1 - \frac{1}{6}\tan x = \tan x + \frac{1}{6} \quad \Leftrightarrow \quad \frac{5}{6} = \frac{7}{6}\tan x.$$

So, $\tan x = (5/6)/(7/6) = 5/7$.

18. Suppose that x^3+px^2+qx+r is a cubic with a double root at a and another root at b, where a and b are real numbers. If $p=-6$ and $q=9$, what is r?

(A) 0

(B) 4

(C) 108

(D) It could be 0 or 4.

(E) It could be 0 or 108.

(F) 18

(G) -4

(H) -108

(I) It could be 0 or -4.

(J) It could be 0 or -108.

(K) It could be 4 or -4.

(L) There is no such value of r.

(M) 1

(N) -2

(O) It could be -2 or -4.

(P) It could be 0 or -2.

(Q) It could be 2007 or a yippy dog.

(R) 2007

Answer: (I) It could be 0 or -4.

Solution: We know $(x^3+px^2+qx+r)=(x-a)^2(x-b)$. Then we have $2a+b=6$ and $a^2+2ab=9$. We are looking for $r=-a^2b$. From the first two equations we get $a^2+2a(6-2a)=9$, so $3a^2-12a+9=0$. This gives $a=1, b=4$ and $a=3, b=0$. Both of these are valid, and they give $r=0,-4$.

19. One day Jason finishes his math homework early, and decides to take a jog through his neighborhood. While jogging, Jason trips over a leprechaun. After dusting himself off and apologizing to the odd little magical creature, Jason, thinking there is nothing unusual about the situation, starts jogging again. Immediately the leprechaun calls out, "hey, stupid, this is your only chance to win gold from a leprechaun!"

Jason, while not particularly greedy, recognizes the value of gold. Thinking about his limited college savings, Jason approaches the leprechaun and asks about the opportunity. The leprechaun hands Jason a fair coin and tells him to flip it as many times as it takes to flip a head. For each tail Jason flips, the leprechaun promises one gold coin.

If Jason flips a head right away, he wins nothing. If he first flips a tail, then a head, he wins one gold coin. If he's lucky and flips ten tails before the first head, he wins *ten gold coins.* What is the expected number of gold coins Jason wins at this game?

(A) 0 (B) $\frac{1}{10}$ (C) $\frac{1}{8}$

(D) $\frac{1}{5}$ (E) $\frac{1}{4}$ (F) $\frac{1}{3}$

(G) $\frac{2}{5}$ (H) $\frac{1}{2}$ (I) $\frac{3}{5}$

(J) $\frac{2}{3}$ (K) $\frac{4}{5}$ (L) 1

(M) $\frac{5}{4}$ (N) $\frac{4}{3}$ (O) $\frac{3}{2}$

(P) 2 (Q) 3 (R) 4

(S) 2007

Answer: (L) 1

Solution: First, we note that for any positive integer n, the probability that the first head occurs on the n^{th} flip is $1/2^n$. So, letting S be the expected number of gold coins Jason wins, we note that

$$S = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2^2} + 2 \cdot \frac{1}{2^3} + 3 \cdot \frac{1}{2^4} + \cdots .$$

The series equal to S is neither arithmetic nor geometric, but has properties of both types of series. We call such series arithmetico-geometric series.

Method 1: We can chop S up into infinitely many geometric series that share a single common ratio:

$$\begin{aligned} S_1 &= \tfrac{1}{2^2} + \tfrac{1}{2^3} + \tfrac{1}{2^4} + \cdots = \tfrac{\frac{1}{4}}{1-\frac{1}{2}} = \tfrac{1}{2}, \\ S_2 &= \qquad\;\; \tfrac{1}{2^3} + \tfrac{1}{2^4} + \cdots = \tfrac{\frac{1}{8}}{1-\frac{1}{2}} = \tfrac{1}{4}, \\ S_3 &= \qquad\qquad\quad \tfrac{1}{2^4} + \cdots = \tfrac{\frac{1}{16}}{1-\frac{1}{2}} = \tfrac{1}{8}, \\ \vdots &= \qquad\qquad\quad\; \vdots \qquad\qquad\qquad \vdots \end{aligned}$$

Adding down these columns, we get a new expression for S:

$$S = S_1 + S_2 + S_3 + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots .$$

Now that we have S expressed as a geometric series, we compute:

$$S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

Method 2: Note that twice S is a series that is very similar to S:

$$2S = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 3 \cdot \frac{1}{2^3} + 4 \cdot \frac{1}{2^4} + \cdots .$$

Subtracting S from this series, we get

$$\begin{aligned} 2S - S &= (1-0) \cdot \frac{1}{2} + (2-1)\frac{1}{2^2} + (3-2)\frac{1}{2^3} + \cdots \\ &= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots \\ &= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1. \end{aligned}$$

That is, $2S - S = S = 1$ after we evaluate the same geometric series we found using Method 1.

Finally, if we said the moral of the story was that leprechauns sometimes give gold to students who finish their math homework early, we'd be lying. Yeah, it's a shame.

20. Find the largest integer n such that $2007^{1024} - 1$ is divisible by 2^n.

(A) 1	(B) 2	(C) 3
(D) 4	(E) 5	(F) 6
(G) 7	(H) 8	(I) 9
(J) 10	(K) 11	(L) 12
(M) 13	(N) 14	(O) 15
(P) 16	(Q) 55	(R) 63
(S) 64	(T) 2007	

Answer: (M) 13

Solution: By repeated use of difference of squares factorization, we see that

$$\begin{aligned} 2007^{1024} - 1 &= (2007^{512}+1)(2007^{512}-1) \\ &= (2007^{512}+1)(2007^{256}+1)(2007^{256}-1) \\ &\vdots \\ &= (2007^{512}+1)(2007^{256}+1)(2007^{128}+1)\cdots(2007+1)(2007-1). \end{aligned}$$

Most of the factors are of the form $2007^{2^k} + 1$, and we note that for $k \geq 1$,

$$2007^{2^k} \equiv (-1)^{2^k} \equiv 1^{2^{k-1}} \equiv 1 \pmod 4.$$

Hence $2007^{2^k} + 1 \equiv 2 \pmod 4$, and includes exactly 1 power of 2 in its prime factorization. There are 9 such factors.

The other two factors are 2008 and 2006, and we note that

$$2008 = 2^3 \cdot 251^1,$$
$$2006 = 2^1 \cdot 17^1 \cdot 59^1.$$

Adding the 4 powers of 2 from these factors to the 9 previous, we see that 2^{13} is the largest power of 2 that divides $2007^{1024} - 1$.

21. James writes down fifteen 1's in a row and randomly writes + or − between each pair of consecutive 1's. One such example is

$$1+1+1-1-1+1-1+1-1-1-1-1+1+1-1.$$

What is the probability that the value of the expression James wrote down is 7?

(A) 0	(B) $\frac{6435}{2^{14}}$	(C) $\frac{6435}{2^{13}}$
(D) $\frac{429}{2^{12}}$	(E) $\frac{429}{2^{11}}$	(F) $\frac{429}{2^{10}}$
(G) $\frac{1}{15}$	(H) $\frac{1}{31}$	(I) $\frac{1}{30}$
(J) $\frac{1}{29}$	(K) $\frac{1001}{2^{15}}$	(L) $\frac{1001}{2^{14}}$
(M) $\frac{1001}{2^{13}}$	(N) $\frac{1}{2^7}$	(O) $\frac{1}{2^{14}}$
(P) $\frac{1}{2^{15}}$	(Q) $\frac{2007}{2^{14}}$	(R) $\frac{2007}{2^{15}}$
(S) $\frac{2007}{2^{2007}}$	(T) $\frac{1}{2007}$	(U) $-\frac{2007}{2^{14}}$

Answer: (L) $\frac{1001}{2^{14}}$

Solution: James wrote down 14 signs, so there are 2^{14} total possibilities. In order to get a total of 7, there must be 4 minus signs and 10 plus signs. So the number of successes is $\binom{14}{4}$. The probability is

$$\frac{\binom{14}{4}}{2^{14}} = \frac{14!}{4!10!2^{14}} = \frac{14 \cdot 13 \cdot 12 \cdot 11}{4!2^{14}}$$
$$= \frac{7 \cdot 13 \cdot 11}{2^{14}} = \frac{1001}{2^{14}}.$$

22. Find the value of c such that the system of equations,

$$|x+y| = 2007,$$
$$|x-y| = c,$$

has exactly two solutions (x, y) in real numbers.

(A)	0	(B)	1	(C)	2
(D)	3	(E)	4	(F)	5
(G)	6	(H)	7	(I)	8
(J)	9	(K)	10	(L)	11
(M)	12	(N)	13	(O)	14
(P)	15	(Q)	16	(R)	17
(S)	18	(T)	223	(U)	678
(V)	2007				

Answer: (A) 0

Solution: We can divide the possibilities into three cases:

- When $c = 0$, we have $x = y$, and the solutions are

$$\left(\frac{2007}{2}, \frac{2007}{2}\right) \quad \text{and} \quad \left(-\frac{2007}{2}, -\frac{2007}{2}\right).$$

- When $c < 0$, there are no solutions to the second equation at all, and therefore no solutions to the system.
- When $c > 0$, there are four solutions corresponding to the four systems of equations,

$$x + y = \pm 2007,$$
$$x - y = \pm c,$$

where all combinations of the $\pm$ are valid.

23. Find the product of the nonreal roots of the equation

$$(x^2 - 3x)^2 + 5(x^2 - 3x) + 6 = 0.$$

(A)	0	(B)	1	(C)	-1
(D)	2	(E)	-2	(F)	3
(G)	-3	(H)	4	(I)	-4
(J)	5	(K)	-5	(L)	6
(M)	-6	(N)	$3+2i$	(O)	$3-2i$
(P)	$\frac{-3+i\sqrt{3}}{2}$	(Q)	8	(R)	-8
(S)	12	(T)	-12	(U)	42
(V)	Ying	(W)	2007		

Answer: (F) 3

Solution: First, let's give a name to the quadratic expression we see repeated in the given equation. We let $A = x^2 - 3x$, so the given equation becomes

$$A^2 + 5A + 6 = 0,$$

which has solutions $A = -2$ and $A = -3$.

Now, we know that the solutions to the original equation are all the solutions x to one of the equations,

$$x^2 - 3x = -2,$$
$$x^2 - 3x = -3.$$

The roots of the first of these quadratic equations are 1 and 2, which are real. For the second quadratic equation, we note that the discriminant is negative:

$$(-3)^2 - 4(1)(3) = 9 - 12 = -3.$$

Thus, the roots are nonreal. We rewrite the equation as $x^2 - 3x + 3 = 0$ and apply Vieta's Formula for the product of the roots to get $3/1 = 3$ as our answer.

24. Let N be the smallest positive integer such that $2008N$ is a perfect square and $2007N$ is a perfect cube. Find the remainder when N is divided by 25.

(A)	0	(B)	1	(C)	2
(D)	3	(E)	4	(F)	5
(G)	6	(H)	7	(I)	8
(J)	9	(K)	10	(L)	11
(M)	12	(N)	13	(O)	14
(P)	15	(Q)	16	(R)	17
(S)	18	(T)	19	(U)	20
(V)	21	(W)	22	(X)	23

Answer: (R) 17

Solution: We construct N using what we know about prime factorizations of the given square and cube. We note that

$$2008N = 2^3 \cdot 251^1 \cdot N,$$
$$2007N = 3^2 \cdot 223^1 \cdot N,$$

where 2, 3, 223, and 251 are all prime.

Since $2008N$ is a perfect square, the exponents in its prime factorization are all even, so

$$N = 2^{2a_1+1} \cdot 3^{2a_2} \cdot 223^{2a_3} \cdot 251^{2a_4+1} \cdot x^2,$$

where x is either 1 or composed entirely of primes other than the four we used. Since $2007N$ is a perfect square, the exponents in its prime factorization are all multiples of 3, so

$$N = 2^{3b_1} \cdot 3^{3b_2+1} \cdot 223^{3b_3+2} \cdot 251^{3b_4} \cdot y^3,$$

where y is either 1 or composed entirely of primes other than the four we used.

Now, we find the smallest possible exponents. The two prime factorizations of N above give us systems of linear congruences to solve, though the least possible exponents are small enough that we can just hunt for them. Also, the smallest N occurs when $x = y = 1$, so

$$N = 2^3 \cdot 3^4 \cdot 223^2 \cdot 251^3.$$

Finally, we find the remainder when N is divided by 25:

$$\begin{aligned} N &= 2^3 \cdot 3^4 \cdot 223^2 \cdot 251^3 \\ &\equiv 2^3 \cdot 3^4 \cdot (-2)^2 \cdot 1^3 \\ &\equiv 8 \cdot 81 \cdot 4 \cdot 1 \\ &\equiv 8 \cdot 6 \cdot 4 \equiv 192 \equiv 17 \pmod{25}. \end{aligned}$$

So, 17 is the remainder.

25. Ted's favorite number is equal to

$$1 \cdot \binom{2007}{1} + 2 \cdot \binom{2007}{2} + 3 \cdot \binom{2007}{3} + \cdots + 2007 \cdot \binom{2007}{2007}.$$

Find the remainder when Ted's favorite number is divided by 25.

(A) 0	(B) 1	(C) 2
(D) 3	(E) 4	(F) 5
(G) 6	(H) 7	(I) 8
(J) 9	(K) 10	(L) 11
(M) 12	(N) 13	(O) 14
(P) 15	(Q) 16	(R) 17
(S) 18	(T) 19	(U) 20
(V) 21	(W) 22	(X) 23
(Y) 24		

Answer: (X) 23

Solution: The identity $\binom{n}{r} = \binom{n}{n-r}$ allows us to take a clever, Gauss-like approach to simplifying Ted's favorite number. Let Ted's favorite number be S, then

$$\begin{aligned} S &= 1 \cdot \binom{2007}{1} + 2 \cdot \binom{2007}{2} + 3 \cdot \binom{2007}{3} + \cdots + 2007 \cdot \binom{2007}{2007} \\ &= 1 \cdot \binom{2007}{2006} + 2 \cdot \binom{2007}{2005} + 3 \cdot \binom{2007}{2004} + \cdots + 2007 \cdot \binom{2007}{0}. \end{aligned}$$

Spinning this last expression around and adding it to the original expression for S, we get

$$2S = 2007 \cdot \binom{2007}{0} + 2007 \cdot \binom{2007}{1} + 2007 \cdot \binom{2007}{2} + \cdots + 2007 \cdot \binom{2007}{2007}.$$

Note that by the Binomial Expansion Theorem,

$$(1+1)^{2007} = 1^{2007} \cdot 1^0 \cdot \binom{2007}{0} + 1^{2006} \cdot 1^1 \cdot \binom{2007}{1} + \cdots + 1^0 \cdot 1^{2007} \cdot \binom{2007}{2007}.$$

The left-hand side is just 2^{2007}, and the right-hand side is

$$\binom{2007}{0} + \binom{2007}{1} + \binom{2007}{2} + \cdots + \binom{2007}{2007}.$$

So,

$$2S = 2007 \cdot 2^{2007}.$$

Dividing by 2, we get $S = 2007 \cdot 2^{2006}$.

Now, we make use of Euler's Theorem. Since 2 and 25 are coprime, and $\phi(25) = 5(5-1) = 20$, we note that $2^{20} \equiv 1 \pmod{25}$. Hence,

$$\begin{aligned} S &= 2007 \cdot 2^{2006} \equiv 2007 \cdot (2^{20})^{100} \cdot 2^6 \\ &\equiv 7 \cdot 1^{100} \cdot 2^6 \equiv 7 \cdot 64 \\ &\equiv 7 \cdot 14 \equiv 98 \equiv 23 \pmod{25}. \end{aligned}$$

Thus, 23 is the remainder when Ted's favorite number is divided by 25.

26. Julie runs a website where she sells university themed clothing. On Monday, she sells thirteen Stanford sweatshirts and nine Harvard sweatshirts for a total of \$370. On Tuesday, she sells nine Stanford sweatshirts and two Harvard sweatshirts for a total of \$180. On Wednesday, she sells twelve Stanford sweatshirts and six Harvard sweatshirts. If Julie didn't change the prices of any items all week, how much money did she take in (total number of dollars) from the sale of Stanford and Harvard sweatshirts on Wednesday?

Answer: 300

Solution 1: We begin by turning the words into math. Let s be the sale price of each Stanford sweatshirt, and let h be the sale price of each Harvard sweatshirt. Then, the sales data from Monday and Tuesday tell us that

$$\begin{aligned} 13s + 9h &= 370, \\ 9s + 2h &= 180. \end{aligned}$$

We can solve this system of equations using substitution or elimination. Here, we use elimination, eliminating the Harvard variable:

$$9(9s + 2h) - 2(13s + 9h) = 9 \cdot 180 - 2 \cdot 370 = 880,$$

which simplifies to $55s = 880$, so $s = 16$. Plugging this value for s into the first equation, we get

$$13 \cdot 16 + 9h = 370 \qquad \Leftrightarrow \qquad 208 + 9h = 370.$$

So, $9h = 162$, and thus $h = 18$.

Thus, on Wednesday, the amount of money Julie took in from the sale of Stanford and Harvard sweatshirts is

$$12s + 6h = 12 \cdot 16 + 6 \cdot 18 = 192 + 108 = 300.$$

Solution 2: Adding the system of equations we got from translating the problem into the language of math, we get $22s + 11h = 550$. The $2:1$ ratio of sweatshirts is the same as the ratio sold on Wednesday, so we multiply our newest equation by $6/11$ to get

$$12s + 6h = \frac{6}{11} \cdot 550 = 6 \cdot 50 = 300.$$

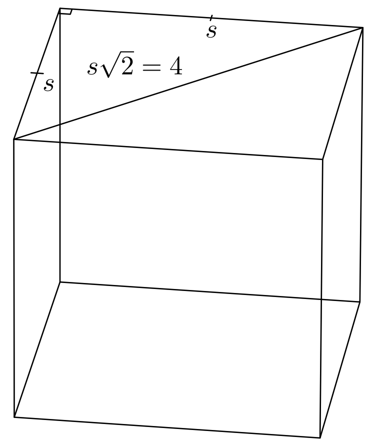


Figure 1: Face diagonal of a cube

27. The face diagonal of a cube has length 4. Find the value of n given that $n\sqrt{2}$ is the *volume* of the cube.

    Answer: 16

    Solution: The diagonal of a (square) face of a cube cuts the square into two isosceles right triangles. If the length of the sides of the square (the lengths of the legs of the right triangles) is s, then by the Pythagorean Theorem,

$$s^2 + s^2 = 4^2,$$

    so $2s^2 = 16$. Then $s^2 = 8$, hence $s = 2\sqrt{2}$.

    In fact it is generally true that the lengths of the sides of an isosceles right triangle are in ratio $1 : 1 : \sqrt{2}$.

    The volume of a cube is the cube of the length of one of its edges. This volume is

$$\left(2\sqrt{2}\right)^3 = 16\sqrt{2} = n\sqrt{2},$$

    so $n = 16$.

28. The space diagonal (interior diagonal) of a cube has length 6. Find the *surface area* of the cube.

    Answer: 72

    Solution: Similarly to the way we use a right triangle to find a relationship between the sides and the diagonals of a square, we use a right triangle to find a relationship between the edges, face diagonals, and space diagonal of a cube. The six faces of a cube lie in six planes, each of which is parallel to one of the other planes, but perpendicular to the others (since the faces meet at right angles).

    Furthermore, each edge is perpendicular to two of the six faces of a cube. We can construct a right triangle using one of these edges, a face diagonal, and a space diagonal of the cube.

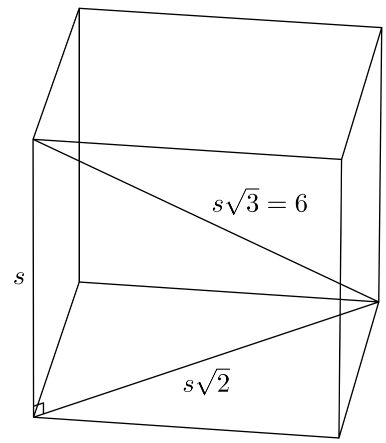


Figure 2: Space diagonal of a cube

Letting s the length of an edge of the cube, we know that each face diagonal has length $s\sqrt{2}$, so by the Pythagorean Theorem,

$$s^2 + \left(s\sqrt{2}\right)^2 = 6^2 \qquad \Leftrightarrow \qquad 3s^2 = 36.$$

Hence $s^2 = 12$, which gives us the area of each square face of the cube. Thus, the total surface area of the cube is $6 \cdot 12 = 72$.

29. Let S be equal to the sum

$$1 + 2 + 3 + \cdots + 2007.$$

Find the remainder when S is divided by 1000.

Answer: 28

Solution 1: The sum of the first 2007 positive integers, like any arithmetic series, can be computed using a "copy" of the series, reversed:

$$\begin{aligned} S &= \quad 1 \quad + \quad 2 \quad + \quad 3 \quad + \cdots + \quad 2007, \\ S &= \ 2007 \ + \ 2006 \ + \ 2005 \ + \cdots + \quad 1, \\ S + S &= (1 + 2007) + (2 + 2006) + (3 + 2005) + \cdots + (2007 + 1). \end{aligned}$$

Adding the first two of the above three equations gave us the third, which is in fact the sum when the number 2008 is added together a total of 2007 times:

$$2S = 2008 + 2008 + 2008 + \cdots + 2008 = 2007 \cdot 2008 = 4030056.$$

Thus, $S = 2015028$, which leaves a remainder of 28 when divided by 1000.

Solution 2: Modular arithmetic allows for simplified computation of the remainder:

$$S = \frac{2007 \cdot 2008}{2} = 2007 \cdot 1004 \equiv 7 \cdot 4 \equiv 28 \pmod{1000}.$$

Solution 3: Some clever students, not knowing a formula for the sum of an arithmetic series, or the modular arithmetic in Solution 2, might pair terms up with no remainder:

$$(1 + 1999) + (2 + 1998) + (3 + 1997) + \cdots + (999 + 1001)$$

$$+1000 + 2000 + 2001 + 2002 + \cdots + 2007.$$

Casting out the multiples of 1000, we have the simple sum $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$.

30. While working with some data for the Iowa City Hospital, James got up to get a drink of water. When he returned, his computer displayed the "blue screen of death" (it had crashed). While rebooting his computer, James remembered that he was nearly done with his calculations since the last time he saved his data. He also kicked himself for not saving before he got up from his desk. He had computed three positive integers a, b, and c, and recalled that their product is 24, but he didn't remember the values of the three integers themselves. What he really needed was their sum. He knows that the sum is an even two-digit integer less than 25 with fewer than 6 divisors. Help James by computing $a + b + c$.

Answer: 10

Solution: We could try listing all possible ordered triples (a, b, c). However, even an organized approach might take a long time, particularly listing possible permutations such as

$$(12, 2, 1), (12, 1, 2), (2, 12, 1), (2, 1, 12), (1, 12, 2), (1, 2, 12).$$

We can work much faster by ignoring the distinction between these different ordered triples. So, we only consider cases where a is at least as large as b, which is in turn at least as large as c.

The mathematically technical way of saying this is "Without loss of generality, we let $a \geq b \geq c$."

Now, since a is a divisor of 24, there are only a few possible cases:

- $a = 24$, in which case $bc = 1$, so $b = c = 1$. In this case $a + b + c = 26$.
- $a = 12$, in which case $bc = 2$, so $b = 2$ and $c = 1$. In this case $a + b + c = 15$.
- $a = 8$, in which case $bc = 3$, so $b = 3$ and $c = 1$. In this case $a + b + c = 12$.
- $a = 6$, in which case $bc = 4$. When $b = 4$ and $c = 1$, then $a+b+c = 11$. When $b = c = 2$, then $a + b + c = 10$.
- $a = 4$, in which case $bc = 6$, so $b = 3$ and $c = 2$. In this case $a + b + c = 9$.
- If $a \leq 3$, then $bc \geq 8$, so the greater of b and c is at least $\sqrt{8} = 2\sqrt{2}$. Since

  $$4 < 8 < 9 \qquad \Leftrightarrow \qquad \sqrt{4} < \sqrt{8} < \sqrt{9},$$

  so $2 < 2\sqrt{2} < 3$. Since b and c are integers, the greater of the two is at least 3. But $3 \nmid 8$, so the greater of b and c is at least 4. This contradicts the fact that $a \geq b \geq c$, so we are done with our casework.

The possible even integer sums are 26, 12, and 10. Only 12 and 10 are less than 25, and 12 has six divisors, while 10 has four, so $a + b + c = 10$.

31. Let x be the length of one side of a triangle and let y be the height to that side. If $x+y = 418$, find the maximum possible *integral value* of the area of the triangle.

Answer: 21840

Solution 1: The area of a triangle is half the product of any base and the altitude to that base. So, $xy/2$ is the area of the triangle. We can maximize the area by maximizing xy. Since $x+y=418$, we have that $y=418-x$, and we can substitute for y:

$$xy = x(418-x) = -x^2 + 418x.$$

Completing the square allows us to maximize this expression:

$$-x^2 + 418x = -(x-209)^2 + 209^2.$$

The value of the squared term is minimal when $x = 209$, and so the minimum possible value of xy occurs when $x = y = 209$ and is 209^2. Therefore, the maximum possible area of the triangle is

$$\frac{209^2}{2} = \frac{43681}{2} = 21840.5.$$

We want the largest possible integral value of the area, and so now we note that, in terms of just x, the area of of the triangle is a continuous quadratic function:

$$f(x) = \frac{-x^2 + 418x}{2}.$$

This graph of this function is just an ordinary downward opening parabola whose range consists of all real number less than or equal to 21840.5. Thus, the maximum integral area is 21840.

Solution 2: By the AM-GM Inequality,

$$\frac{x+y}{2} \geq \sqrt{xy} \qquad \Leftrightarrow \qquad 209 \geq \sqrt{xy},$$

with equality if and only if $x = y$. Squaring the second inequality, we get

$$209^2 \geq xy \qquad \Leftrightarrow \qquad \frac{209^2}{2} \geq \frac{xy}{2}.$$

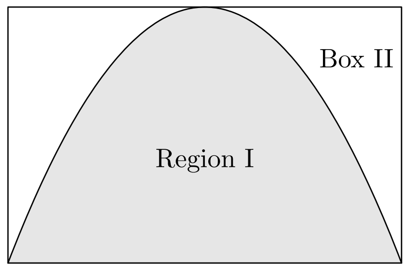
The equality condition for AM-GM requires that $x = y$, which gives us a maximum area of $209^2/2 = 21840.5$. We finish by noting that the area of the triangle can be expressed as a continuous function of x (as in Solution 1), so the range of possible areas is $(0, 21840.5)$, and the maximum integral area is $\lfloor 21840.5 \rfloor = 21840$.

32. When a rectangle frames a parabola such that a side of the rectangle is parallel to the parabola's axis of symmetry, the parabola divides the rectangle into regions whose areas are in the ratio 2 to 1. How many integer values of k are there such that $0 < k \leq 2007$ and the area between the parabola $y = k - x^2$ and the x-axis is an integer?

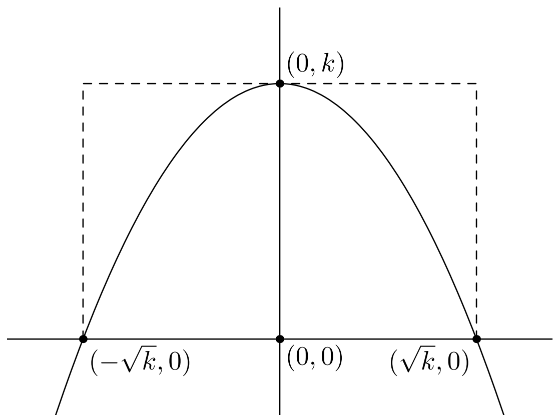
    Answer: 14

    Solution: The highest point on the parabola occurs when $x = 0$ and $y = k - 0^2 = k$. We say the vertex is at $(0, k)$. Computing the x-intercepts will allow us to frame the parabola with a rectangle:

$$0 = k - x^2 \qquad \Leftrightarrow \qquad x^2 = k \qquad \Leftrightarrow \qquad x = \pm\sqrt{k}.$$



area(I) = $\frac{2}{3}$ area(II)



The dimensions of the rectangle are k and $2\sqrt{k}$, so its area is $2k^{\frac{3}{2}}$. The area under the parabola is thus

$$\frac{2}{3} \cdot 2k^{\frac{3}{2}} = \frac{4}{3}k^{\frac{3}{2}}.$$

First, the area is never an integer when $\sqrt{k}$ is not rational, so k must be a perfect square.

Letting $k = a^2$, for some integer a, allows us to proceed without messy radicals:

$$\frac{4}{3}k^{\frac{3}{2}} = \frac{4}{3}(a^2)^{\frac{3}{2}} = \frac{4}{3}a^3.$$

This value is an integer if and only if a is a multiple of 3. This means that k must be a perfect square multiple of 3.

Since $0 < k \leq 2007$, we note that $44 < \sqrt{2007} < 45$, so the possible values of k are the first row of the correspondence grid below:

3^2	6^2	9^2	...	39^2	42^2
1	2	3	...	13	14

We count in the second row, and the answer is 14.

33. How many *odd* four-digit integers have the property that their digits, read left to right, are in strictly decreasing order?

Answer: 80

Solution: An integer with the given property has a units digit that is at most 6, because the three digits to its left are distinct and larger. Since we are counting such integers that are odd, there are three cases: the units digit is 1, the units digit is 3, or the units digit is 5.

We consider the three cases:

- First, we consider the case in which the units digit is 1. The three digits to the left of the units digit are all greater than 1, and there are 8 such possible digits. In fact for each set of three distinct digits greater than 1, there is exactly one such integer that fits the given property. For instance, if the three digits greater than 1 are 2, 4, and 7, then the four-digit odd integer with the given property is 7421. The total number of such combinations of three digits greater than 1 is

$$\binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3!} = 56.$$

- Similarly, the total number of such four-digit integers with units digit 3 is $\binom{6}{3} = 20$, and
- the total number of such four-digit integers with units digit 5 is $\binom{4}{3} = 4$.

In total, there are $56 + 20 + 4 = 80$ odd four-digit integers whose digits are in strictly descreasing order.

34. Let a/b be the probability that a randomly selected divisor of 2007 is a multiple of 3. If a and b are relatively prime positive integers, find $a + b$.

Answer: 5

Solution 1: Prime factorization tends to be the foundation of most problems involving divisors, so we start there:

$$2007 = 3^2 \cdot 223^1.$$

A divisor of 2007 is of the form

$$\pm 3^a \cdot 223^b,$$

where $0 \leq a \leq 2$ and $0 \leq b \leq 1$.

Now, we compute the total number of divisors by multiplying 2 choices for the signs, 3 choices for a, and 2 choices for b to get $2 \cdot 3 \cdot 2 = 12$. The multiples of 3 are the ones in which $a > 0$, and there are $2 \cdot 2 \cdot 2 = 8$. So, the probability that a randomly selected divisor of 2007 is a multiple of 3 is $8/12 = 2/3$, and the answer is $2 + 3 = 5$.

Note: the answer is the same whether or not negative divisors are taken into account.

Solution 2: Again, the divisors are of the form

$$\pm 3^a \cdot 223^b.$$

Since the values of a and b are *independent*, we only need to think about the values of a, which must be at least 1 in order for a divisor to be a multiple of 3. There are 3 possible values of a, of which 2 are at least 1, so the probability is $2/3$, and again the answer is $2 + 3 = 5$.

35. Find the greatest natural number possessing the property that each of its digits except the first and last one is less than the arithmetic mean of the two neighboring digits.

Answer: 96,433,469

Credit: This problem appeared on the grade 9 Leningrad Mathematical Olympiad in 1987, which I felt was both old enough and obscure enough to replicate. It may after all be a classic

– it should be! In particular, this problem was included because of its surprising relationships to so many elementary mathematical concepts.

Solution: There are several reasonable angles of approach for this problem. Before we get to anything solid, let's play around and construct a few integers with the given property. We'll start small and build up, and make note of some properties that we notice along the way. Often, it's one of these properties, or a combination of several that lead us to a certain solution:

989	9889	9778	97679	86568	86569	865569	7421247
979	9779	9766	96569	85458	85459	854458	9643469
969	9669	7669	95259	84148	84126	841249	96433469

This is certainly not an exhaustive list! We could of course snip digits from the front or end of any of these integers to create others of shorter length:

$$96433469 \rightarrow 9643346 \rightarrow 964334 \rightarrow 64334 \rightarrow \ldots$$

There are many other such integers as well.

Many students might simply notice that constructing the longest possible integer with these properties (most digits) helps us find the answer. The digits can go down then up:

$$9 - 6 - 4 - 3 - 3 - 4 - 6 - 9$$

The key is in noting (and better yet establishing) that the *rate of change* of the movement from digit to digit gets lower (from left to right). Look at the differences between the digits of some of the numbers in our list above:

```
9     7     7     8
   +2    0    −1

7     4     2     1     2     4     7
   +3    +2    +1    −1    −2    −3

9     6     4     3     3     4     6     9
   +3    +2    +1    0     −1    −2    −3
```

Pictorially, we can see parabolic and nearly-parabolic representations of many of these numbers, particularly the largest one (do you see why?):

```
9 6 4 3 3 4 6 9
*             *
*             *
*             *
* *         * *
* *         * *
* * *     * * *
* * * * * * * *
* * * * * * * *
* * * * * * * *
```

Finally, we note what some highly experienced solvers might have noted more explicitly from the start: Letting A be an integer with the given description, with decimal representation $\overline{a_1 a_2 a_3 \ldots a_n}$, then

$$a_{i+1} < \frac{a_i + a_{i+2}}{2} \qquad \Leftrightarrow \qquad a_i - a_{i+1} > a_{i+1} - a_{i+2},$$

where the latter inequality is just the definition of an arithmetic progression where the '=' sign is replaced with a '>' sign.

Now we want to construct the largest possible such integer. The digits will go down before going up (if we are to use the maximal number of digits), so we start with 9. The digits will get lower to a point. The digits can only get lower three total times because getting lower four total times requires that the difference between the first and fifth digits be at least

$$|(-4) + (-3) + (-2) + (-1)| = 10,$$

which is impossible with *decimal* digits.

There are still multiple possibilities such as 96433469 and 95211247, but starting with the lowest difference between the earliest digits allows us to maintain the largest possible integer, so the left-to-right differences we use are $3, 2, 1, 0, -1, -2, -3$ in that order. These give us the integer 96433469.

Follow-up problem: How would the answer change if we allowed the integers to be written in other number bases?

36. Let b be a real number randomly selected from the interval $[-17, 17]$. Then, m and n are two relatively prime positive integers such that m/n is the probability that the equation

$$x^4 + 25b^2 = (4b^2 - 10b)x^2$$

has *at least* two distinct real solutions. Find the value of $m + n$.

Answer: 63

Solution: Noting that $10bx^2$ is twice the geometric mean of x^4 and $25b^2$, we reorganize the given equation as a difference of squares:

$$x^4 + 10bx^2 + 25b^2 - 4b^2x^2 = 0 \qquad \Leftrightarrow \qquad (x^2 + 5b)^2 - (2bx)^2 = 0.$$

Thus,

$$[(x^2 + 5b) + 2bx][(x^2 + 5b) - 2bx] = 0 \qquad \Leftrightarrow \qquad (x^2 + 2bx + 5b)(x^2 - 2bx + 5b) = 0.$$

So, one of the two following quadratic equations must hold:

$$x^2 + 2bx + 5b = 0,$$
$$x^2 - 2bx + 5b = 0.$$

Then, by the quadratic formula, we have four possible solutions:

$$x = b \pm \sqrt{b^2 - 5b} \qquad \text{or} \qquad x = -b \pm \sqrt{b^2 - 5b}.$$

Now, we note that all four of these expressions describe real numbers if and only if the [common] expression under the radical is nonnegative:

$$b^2 - 5b \geq 0 \qquad \Leftrightarrow \qquad b(b-5) \geq 0.$$

This is true unless b and $b-5$ have opposite signs. In other words, the inequality holds unless $0 < b < 5$.

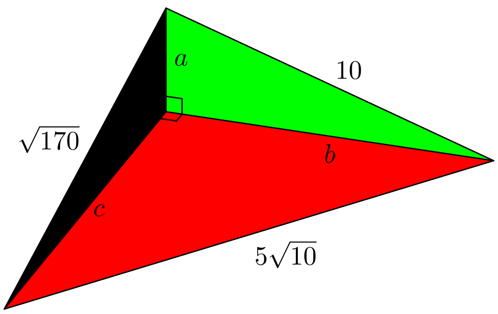
Now, we note that all four roots are the same if $b = 0$. Otherwise, we note that

$$b + \sqrt{b^2 - 5b} \neq -b + \sqrt{b^2 - 5b},$$

making two of the roots distinct. So, the roots are distinct when b is within $[-17, 0)$ or b is within $[5, 17]$. The sizes of these intervals are 17 and 12, and the size of the given interval is 34, so the probability we are looking for is

$$\frac{12+17}{34} = \frac{29}{34} = \frac{m}{n},$$

where $m = 29$ and $n = 34$, so $m + n = 63$.



37. Rob is helping to build the set for a school play. For one scene, he needs to build a multi-colored tetrahedron out of cloth and bamboo. He begins by fitting three lengths of bamboo together, such that they meet at the same point, and each pair of bamboo rods meet at a right angle. Three more lengths of bamboo are then cut to connect the other ends of the first three rods. Rob then cuts out four triangular pieces of fabric: a blue piece, a red piece, a green piece, and a yellow piece. These triangular pieces of fabric just fill in the triangular spaces between the bamboo, making up the four faces of the tetrahedron. The areas in square feet of the red, yellow, and green pieces are 60, 20, and 15 respectively. If the blue piece is the largest of the four sides, find the number of square feet in its area.

    Answer: 65

    Solution: First, we draw the tetrahedron, as above. Labeling the sides, we can now build equations relating the lengths of the sides to the areas of the three right triangles:

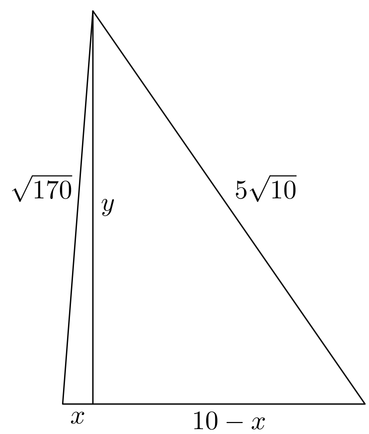
$$\frac{ab}{2} = 15 \quad \Leftrightarrow \quad ab = 30,$$
$$\frac{ac}{2} = 20 \quad \Leftrightarrow \quad ac = 40,$$
$$\frac{bc}{2} = 60 \quad \Leftrightarrow \quad bc = 120.$$

    We can solve this system of equations (on the right, after we eliminated the fractions) by multiplying them together, then using the result:

$$(ab)(ac)(bc) = (30)(40)(120) \quad \Leftrightarrow \quad a^2b^2c^2 = 12^2 \cdot 10^3,$$

    so $abc = 12 \cdot 10\sqrt{10} = 120\sqrt{10}$. Dividing this equation by the three equations above, we get $a = \sqrt{10}$, $b = 3\sqrt{10}$, and $c = 4\sqrt{10}$.

    Now, there are numerous ways in which we could find the area of the large triangle from this point. The volume of the tetrahedron and the altitude to the largest face would allow us to determine the area of the face. Coordinates might help in that regard. The vertex where the right angles meet could represent the origin, with the edges of the tetrahedron moving along the axes. Such a model simplifies much of the computation. We can also find the lengths of



the sides of the larger triangle:

$$\sqrt{(\sqrt{10})^2 + (3\sqrt{10})^2} = 10,$$
$$\sqrt{(\sqrt{10})^2 + (4\sqrt{10})^2} = \sqrt{170},$$
$$\sqrt{(3\sqrt{10})^2 + (4\sqrt{10})^2} = 5\sqrt{10}.$$

A few brave students, undaunted by messy computation, might employ Heron's Formula at this point, though we recommend taking the time to hunt for a nicer approach.

Dropping an altitude to the side of integer length (again trying to avoid messy computation), we set variables equal to the unknown lengths as in the second diagram.

Applying the Pythagorean Theorem twice gives us a couple of equations that look somewhat alike:

$$x^2 + y^2 = 170,$$
$$x^2 - 20x + 100 + y^2 = 250.$$

Subtracting the second equation from the first yields

$$20x - 100 = -80 \qquad \Leftrightarrow \qquad 20x = 20 \qquad \Leftrightarrow \qquad x = 1.$$

Plugging this value back into either equation (the first is easiest), we get

$$1 + y^2 = 170 \qquad \Leftrightarrow \qquad y^2 = 169 \qquad \Leftrightarrow \qquad y = \pm 13.$$

The positive value is the altitude, so the area of the largest triangle is $(10 \cdot 13)/2 = 65$.

Investigation: Note that $15^2 + 20^2 + 60^2 = 65^2$. Must the sum of the squares of the areas of three of the sides be equal to the square of the area of largest side?

38. Find the largest positive integer that is equal to the cube of the sum of its digits.

Answer: 19683

Solution: Let N be the integer we seek, let d be the number of digits of N, and let $S(N)$ be the sum of the digits of N. Since $S(N) \leq 9d$ and $N \geq 10^{d-1}$, we note that

$$9^3 d^3 \geq [S(N)]^3 = N \geq 10^{d-1}.$$

From the outer parts of this inequality, $729d^3 \geq 10^{d-1}$, where the right side of the inequality grows much faster than the left as d ranges through the positive integers. In particular, we quickly find that when $d = 7$,

$$9 \cdot 9 \cdot 9 \cdot 7 \cdot 7 \cdot 7 < 10^6,$$

and the inequality is violated. A simple induction argument shows us that the inequality is violated for larger values of d as well.

If $d = 6$, then $S(N) \leq 54$, so

$$[S(N)]^3 \leq 54^3 = 157464 \qquad \Rightarrow \qquad N \leq 157464.$$

Note that $S(N) < S(199999) = 46$, thus

$$[S(N)]^3 \leq 45^3 = 91125 \qquad \Rightarrow \qquad N \leq 91125,$$

which contradicts the fact that $d = 6$, so we know that N has at most 5 digits.

Now we search to see if there is at least one value of N with 5 digits that satisfies the desired property. We can place loose bounds on $S(N)$:

$$8000 < N < 125000 \qquad \Leftrightarrow \qquad 8000 < [S(N)]^3 < 125000.$$

Taking the cube roots of the last inequality, we have $20 < [S(N)] < 50$.

We could at this point start cubing potential digit sums, but a little modular arithmetic additionally simplifies our task. More general than the divisibility rule for 9 is the fact that

$$m \equiv S(m) \pmod 9$$

for all positive integers m (a proof of this fact involves writing the value of m according to the place values of its digits). Now, since $N = [S(N)]^3$, we have that

$$N \equiv S(N) \equiv [S(N)]^3 \pmod 9.$$

So,

$$[S(N)]^3 - S(N) \equiv 0 \pmod 9 \qquad \Leftrightarrow \qquad S(N)[S(N)+1][S(N)-1] \equiv 0 \pmod 9.$$

Since $S(N)-1, S(N)$, and $S(N)+1$ are distinct modulo 3, we know that exactly one of them is a multiple of 9. This leaves us with just a few possible values of $S(N)$ that we now cube:

$$\begin{aligned}
26^3 &= 17576 && \rightarrow && S(17576) = 26,\\
27^3 &= 19683 && \rightarrow && S(19683) = 27,\\
28^3 &= 21952 && \rightarrow && S(21952) = 19,\\
35^3 &= 42875 && \rightarrow && S(42875) = 26,\\
36^3 &= 46656 && \rightarrow && S(46656) = 27,\\
37^3 &= 50653 && \rightarrow && S(50653) = 19,\\
44^3 &= 85184 && \rightarrow && S(85184) = 26,\\
45^3 &= 91125 && \rightarrow && S(91125) = 18,\\
46^3 &= 97336 && \rightarrow && S(97336) = 28.
\end{aligned}$$

Thus we see that $27^3 = 19683$ is the largest positive integer that is the cube of the sum of its digits.

39. Let a and b be relatively prime positive integers such that a/b is the sum of the real solutions to the equation
$$\sqrt[3]{3x-4} + \sqrt[3]{5x-6} = \sqrt[3]{x-2} + \sqrt[3]{7x-8}.$$
Find $a+b$.

Answer: 13

Solution: All of the numbers under the radicals are in the form $(2n-1)x - 2n$, but more importantly, the sums of the linear functions under the radicals on each side of the equation are the same:
$$(3x-4) + (5x-6) = (x-2) + (7x-8).$$
Letting $a = \sqrt[3]{3x-4}$, $b = \sqrt[3]{5x-6}$, $c = \sqrt[3]{x-2}$, and $d = \sqrt[3]{7x-8}$, we have a system of equations:
$$a + b = c + d,$$
$$a^3 + b^3 = c^3 + d^3.$$
Cubing the first equation gives us an equation with a lot of terms common to the second equation:
$$a^3 + 3a^2b + 3ab^2 + b^3 = c^3 + 3c^2d + 3cd^2 + d^3.$$
Subtracting this new equation from the second equation in the system, we have
$$3a^2b + 3ab^2 = 3c^2d + 3cd^2 \qquad \Leftrightarrow \qquad ab(a+b) = cd(c+d).$$
Either $a + b = c + d = 0$, or else we can divide the last equation by $a + b = c + d$ to get $ab = cd$.

- If $a + b = c + d = 0$, then $a = -b$ and $a^3 = -b^3$. That is,
$$3x - 4 = -(5x - 6) \qquad \Leftrightarrow \qquad 8x - 10 = 0,$$
so $x = 5/4$.
- If $ab = cd$, then
$$\sqrt[3]{((3x-4)(5x-6)} = \sqrt[3]{(x-2)(7x-8)} \qquad \Leftrightarrow \qquad (3x-4)(5x-6) = (x-2)(7x-8).$$
Expanding and solving, we have
$$15x^2 - 38x + 24 = 7x^2 - 22x + 16 \qquad \Leftrightarrow \qquad 8x^2 - 16x + 8 = 0,$$
so $8(x-1)^2 = 0$, meaning $x = 1$.

The sum of the solutions is $5/4 + 1 = 9/4$, so the answer is $4 + 9 = 13$.

40. Let S be the sum of all x such that $1 \leq x \leq 99$ and

$$\{x^2\} = \{x\}^2.$$

Compute $\lfloor S \rfloor$.

Answer: 641999

Inspiration: This problem was inspired by the 1981 ARML Power Question.

Solution: As in many problems involving integer and fraction parts, we begin by chopping the variable up into integer and fraction parts. Let $x = m + a$, where $\lfloor x \rfloor = m$ and $\{x\} = a$. Then,

$$\{x^2\} = \{(m+a)^2\} = \{m^2 + 2am + a^2\}.$$

Now, we note that

$$\{x\}^2 = a^2.$$

At this point, we note that the fraction parts of two real numbers are equal if and only if the difference between those two real numbers is an integer. See if you can formally prove this theorem that we now make handy:

$$\{x^2\} = \{x\}^2 \qquad \Rightarrow \qquad (m^2 + 2am + a^2) - a^2 = k,$$

for some integer k. Hence $m^2 + 2am$ is an integer. Since m^2 itself is an integer, we know that $(m^2 + 2am) - m^2 = 2am$ is an integer.

Let j be an integer such that $j = 2am$, then $a = j/2m$. Now we can find solutions by intervals of unit size. For instance, when $m = 1$, we have two solutions, where $j = 0$ and $j = 1$:

$$x = 1 + 0/2(1) = 1, \quad x = 1 + 1/2(1) = 3/2.$$

For $m = 2$, we have solutions for integers $0 \leq j < 2(2)$:

$$x = 2 + 0/2(2) = 2, \quad x = 2 + 1/2(2) = 9/4, \quad x = 2 + 2/2(2) = 5/2, \quad x = 2 + 3/2(2) = 11/4.$$

In general, for $m \leq x < m + 1$, there are $2m$ solutions corresponding to the integers $0 \leq j < 2m$.

Let $T(m)$ be the sum of the $2m$ solutions for x where $m \leq x < m + 1$ for a positive integer m. So, our goal is to compute

$$S = \left(\sum_{m=1}^{98} T(m) \right) + 99,$$

where the $+99$ at the end is due to the fact that $x = 99$ is a solution (as are all integers, which should be clear from our previous work).

Now we find a closed form for $T(m)$. The $2m$ solutions m, $m + 1/2m$, $m + 2/2m$, ..., $m + (2m-1)/2m$ are in arithmetic progression, so we can sum them easily. Their average is the average of the smallest and the largest solutions:

$$\frac{1}{2}\left(2m + \frac{2m-1}{2m}\right) = m + \frac{2m-1}{4m}.$$

So, the sum of all $2m$ of these solutions is

$$2m\left(m + \frac{2m-1}{4m}\right) = 2m^2 + m - \frac{1}{2}.$$

Now, we jump into computing S:

$$\begin{aligned} S &= 99 + \sum_{m=1}^{98} T(m) \\ &= 99 + \sum_{m=1}^{98} \left(2m^2 + m - \frac{1}{2}\right) \\ &= 99 + 2\sum_{m=1}^{98} m^2 + \sum_{m=1}^{98} m - \sum_{m=1}^{98} \frac{1}{2} \\ &= 99 + 2 \cdot \frac{98 \cdot 99 \cdot 197}{6} + \frac{98 \cdot 99}{2} - 49 \\ &= 99 + 637098 + 4851 - 49 = 641999. \end{aligned}$$

And then of course, $\lfloor S \rfloor = \lfloor 641999 \rfloor = 641999$.

41. The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the 10^nth digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100^{th} digit enters the sequence in the placement of the two-digit integer 55. Find the value of $f(2007)$.

Answer: 2004

Credit: A version of this problem appeared on the 1987 Putnam Exam, requiring proof of the result, which is certainly a more difficult task than computing the answer.

Solution: A direct approach to this problem is made difficult due to the unusual definition of $f(n)$. Easier than trying to find a formula for $f(n)$ is noting the number of digits that it takes to write the $10^k - 1$ smallest natural numbers (those with k or fewer digits) for positive integers k.

- Writing the 9 1-digit positive integers requires $9 \cdot 1 = 9$ digits.
- Writing the 90 2-digit positive integers requires $90 \cdot 2 = 180$ digits.
- Writing the 900 3-digit positive integers requires $900 \cdot 3 = 2700$ digits.

In general, writing the $10^k - 10^{k-1}$ k-digit positive integers requires using $k(10^k - 10^{k-1})$ digits. So, we let $T(k)$ be the number of digits it takes to write the $10^k - 1$ smallest natural numbers (those with k or fewer digits), and note that

$$T(k) = 9 + 180 + 2700 + \cdots + k(10^k - 10^{k-1}) = \sum_{j=1}^{k} j(10^j - 10^{j-1}).$$

Now we note that $f(2007) = m$ when

$$T(m-1) < 10^{2007} \leq T(m).$$

We also note that by far the largest term in the summation of T is the last term. The greater than geometric rate of growth allows us to estimate T using the last term alone. For instance,

$$T(m) = \sum_{j=1}^{m} j(10^j - 10^{j-1}) \approx m \cdot 9 \cdot 10^{m-1}.$$

Letting this quantity equal 10^{2007}, we get

$$m \cdot 9 \cdot 10^{m-1} \approx 10^{2007} \qquad \Leftrightarrow \qquad 9m \approx 10^{2008-m}.$$

Isolating m on the left-hand side of the last equation, we get

$$m \approx 10^{2008-m}/9 \approx 10^{2007-m}.$$

Obviously, $2007 - m$ can't be much larger than 0, and we find that when $2007 - m = 3$, we get the closest approximation. That is, we conjecture that $f(2007) = m = 2004$.

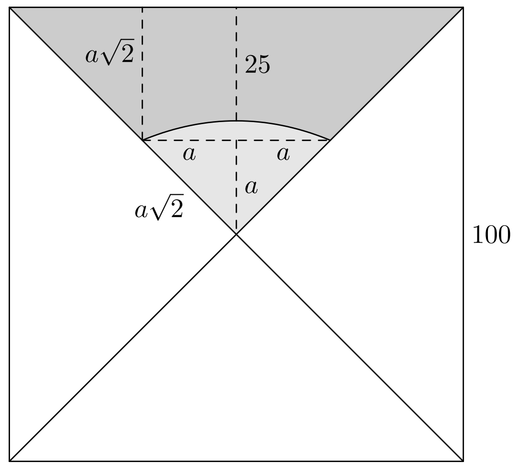
We prove our conjecture using boundaries that are tighter and easier to compute than $T(2004)$. First, we use a geometric series to show that $f(2007) > 2003$:

$$\begin{aligned}
T(2003) = \sum_{j=1}^{2003} j(10^j - 10^{j-1}) \quad &< \quad \sum_{j=1}^{2003} 2003(10^j - 10^{j-1}) \\
&= \quad \sum_{j=1}^{2003} 2003 \cdot 10^{j-1} \cdot (10-1) = \sum_{j=1}^{2003} 9 \cdot 2003 \cdot 10^{j-1} \\
&= \quad \sum_{j=1}^{2003} 18027 \cdot 10^{j-1} = 18027 \sum_{j=1}^{2003} 10^{j-1} \\
&= \quad 18027(1 + 10 + 100 + \cdots + 10^{2002}) \\
&= \quad 18027 \cdot \frac{10^{2003} - 1}{9} \\
&= \quad 2003(10^{2003} - 1) < 10^4 \cdot 10^{2003} = 10^{2007}.
\end{aligned}$$

Now, we use the largest term of $T(2004)$ to show that $f(2007) \leq 2004$:

$$T(2004) = \sum_{j=1}^{2004} j(10^j - 10^{j-1}) > 2004(10^{2004} - 10^{2003}) = 18036 \cdot 10^{2003} > 10^{2007}.$$

Since $T(2003) < 10^{2007} \leq T(2004)$, we have that $f(2007) = 2004$.



42. During a movie shoot, a stuntman jumps out of a plane and parachutes to safety within a 100 foot by 100 foot square field, which is entirely surrounded by a wooden fence. There is a flag pole in the middle of the square field. Assuming the stuntman is equally likely to land on any point in the field, the probability that he lands closer to the fence than to the flag pole can be written in simplest terms as

$$\frac{a - b\sqrt{c}}{d},$$

where all four variables are positive integers, c is a multiple of no perfect square greater than 1, a is coprime with d, and b is coprime with d. Find the value of $a + b + c + d$.

Answer: 17

Solution: We divide the field into four symmetric regions, such that each region consists of the points closest to a particular side of the triangle, as in the diagram.

Within one of the regions, the locus of points that are closer to the fence than the flag pole is bounded by the parabola which traces the locus of points that are equidistant from the fence and the flag pole. Since the regions are symmetric, the probability that the stuntman lands closer the fence than the flag pole is the ratio of the area between the parabola and fence in one of these regions to the area of one of these regions:

$$p = \frac{\text{Area between parabola and fence}}{\frac{1}{4} \cdot 100^2} = \frac{\text{Area between parabola and fence}}{2500}.$$

Now, since we know how to find the area under a parabola, we seek to compute $1 - p$ instead of p. This technique is often referred to as *complementary probability*. In the diagram, we see that the verticle distance covered by the value of a and $a\sqrt{2}$ is equal to 50. So,

$$a(\sqrt{2} + 1) = 50 \qquad \Leftrightarrow \qquad a = 50(\sqrt{2} - 1).$$

The width of the parabola is $2a$, and its height is $25 - a$, so the area of the "parabolic sliver" is

$$\frac{2}{3} \cdot 2a(25 - a).$$

The area of the triangle below it is a^2, and so

$$\begin{aligned}
1-p &= \frac{\frac{2}{3}\cdot 2a(25-a)+a^2}{2500} \\
&= \frac{\frac{2a}{3}\cdot(50-2a)+a^2}{50^2} \\
&= \frac{\frac{2\cdot 50(\sqrt{2}-1)}{3}\cdot[50-2\cdot 50(\sqrt{2}-1)]+[50(\sqrt{2}-1)]^2}{50^2} \\
&= \frac{2(\sqrt{2}-1)}{3}\cdot[1-2(\sqrt{2}-1)]+(\sqrt{2}-1)^2 \\
&= \frac{2}{3}(\sqrt{2}-1)(3-2\sqrt{2})+(3-2\sqrt{2}) \\
&= \frac{-5+4\sqrt{2}}{3}.
\end{aligned}$$

Hence,

$$p = 1-\frac{-5+4\sqrt{2}}{3} = \frac{8-4\sqrt{2}}{3} = \frac{a-b\sqrt{c}}{d},$$

where $(a,b,c,d)=(8,4,2,3)$ and $a+b+c+d=17$.

43. Bored of working on her computational linguistics thesis, Erin enters some three-digit integers into a spreadsheet, then manipulates the cells a bit until her spreadsheet calculates each of the following 100 9-digit integers:

$$\begin{gathered}
700\cdot 712\cdot 718+320, \\
701\cdot 713\cdot 719+320, \\
702\cdot 714\cdot 720+320, \\
\vdots \\
798\cdot 810\cdot 816+320, \\
799\cdot 811\cdot 817+320.
\end{gathered}$$

She notes that two of them have exactly 8 positive divisors each. Find the common prime divisor of those two integers.

Answer: 751

Solution: When investigating problems, one of the most important questions that can guide your inner dialogue is as follows:

What's *interesting* about this problem?

In this case, the interesting feature is the common structure of the 100 integers that Erin chose. Each is of the form

$$x(x+12)(x+18)+320 = x^3+30x+216x+320,$$

where x is an integer between 700 and 799 inclusive.

Now, there are several ways to check and see if this polynomial factors, including an application of the Rational Roots Theorem. After all, we're hoping to factor this cubic over integers (if this is not obvious, you should see why shortly). Descartes Rule of Signs also helps us to note that all real roots are negative (there are no sign changes). After a little hunting, we find that

$$x(x+12)(x+18)+320 = x^3+30x+216x+320 = (x+2)(x+8)(x+20).$$

Given that $700 \leq x \leq 799$, each of the factors $x+2$, $x+8$, and $x+20$ is a three-digit integer. If any one of them is composite, then their product must have more than 8 positive divisors. In order to see why, note that the number of positive divisors of a natural number

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots$$

is

$$t(n) = (e_1+1)(e_2+1)(e_3+1)\cdots.$$

The three three-digit integers $x+2$, $x+8$, and $x+20$ must each have a prime divisor not shared by the other two. After all, by the Euclidean Algorithm, their pairwise GCDs are no larger than their pairwise differences, which are 6, 12, and 18. This means three e_i are at least 1, making $t(n) \geq 8$. In order for $t(n) = 8$, we must have $e_1 = e_2 = e_3 = 1$, with $e_i = 0$ for $i \geq 4$. This means each $x+2$, $x+8$, and $x+20$ must be prime.

We could use an approach similar to the Sieve of Eratosthenes to rule out most of the integers from 702 to 819 as prime. We could also simply consult a table of primes to find that 733, 739, 751, 757, and 769 are all prime:

$$\begin{aligned} 731 \cdot 743 \cdot 749 + 320 &= 733 \cdot 739 \cdot 751, \\ 749 \cdot 761 \cdot 767 + 320 &= 751 \cdot 757 \cdot 769. \end{aligned}$$

The common prime divisor amongst these nine-digit integers (which we choose not to multiply out) is 751.

44. A positive integer n between 1 and $N = 2007^{2007}$ inclusive is selected at random. If a and b are natural numbers such that a/b is the probability that N and $n^3 - 36n$ are relatively prime, find the value of $a+b$.

Answer: 1109

Solution: First, we note the prime factorization of N:

$$2007^{2007} = 3^{4014} \cdot 223^{2007}.$$

Next, we note that $f(n) = n^3 - 36n = n(n+6)(n-6)$.

Let $S = \{n : 1 \leq n \leq N\}$. Now, we note each of the following:

- N and $f(n)$ are relatively prime if and only if n is *not* an element of the set

$$A = \{n : n \equiv 0 \pmod 3\}.$$

- N and $f(n)$ are relatively prime if and only if n is *not* an element of the set

$$B = \{n : n \equiv \pm 6 \text{ or } 0 \pmod{223}\}.$$

In summary, we note that N and $f(n)$ are relatively prime if and only if n is *not* an element of the set $A \cup B$. So, we apply the Principle of Inclusion-Exlusion, but first, we note that $A \cap B$ has exactly 3 solutions modulo 669. Now,

$$\begin{aligned} |S| - |S(A \cup B)| &= N - |A| - |B| + \frac{3}{669}N \\ &= N - \frac{1}{3}N - \frac{3}{223}N + \frac{3}{669}N \\ &= \frac{669}{669}N - \frac{223}{669}N - \frac{9}{669}N + \frac{3}{669}N = \frac{440}{669}N. \end{aligned}$$

Hence, the probability that N and $f(n)$ are relatively prime is

$$\frac{|S| - |S(A \cup B)|}{|S|} = \frac{\frac{440}{669}N}{N} = \frac{440}{669} = \frac{a}{b},$$

so $a + b = 1109$.

Note the existence of a more intuitive solution. The probability that n is not an element of A is $2/3$. The probability that n is not an element of B is $220/223$. An understanding of solutions to systems of linear equations includes the fact that these probabilities are independent, so the probability is simply $(2/3)(220/223) = 440/669$.

45. Find the sum of all positive integers B such that $(111)_B = (aabbcc)_6$, where a, b, c represent distinct base 6 digits, $a \neq 0$.

Answer: 237

Credit: This was problem 1750 of the *Crux Mathematicorum*.

Solution: Expanding the given equation algebraically, we get

$$B^2 + B + 1 = a \cdot 6^5 + a \cdot 6^4 + b \cdot 6^3 + b \cdot 6^2 + c \cdot 6^1 + c \cdot 6^0,$$

which can be simplified a bit to

$$B^2 + B + 1 = 6^4 \cdot 7a + 6^2 \cdot 7b + 7c.$$

The right-hand-side of this equation is a multiple of 7, so we can check through the modulo 7 residues of B to see that either $B \equiv 2 \pmod 7$ or $B \equiv 4 \pmod 7$. So, either $B = 7n + 2$ for some integer n, or $B = 7n + 4$ for some integer n.

Substituting in our two possible expressions for B, expanding, and dividing both sides by 7, we have two possible equations:

$$7n^2 + 5n + 1 = 1296a + 36b + c, \tag{3}$$

or

$$7n^2 + 9n + 3 = 1296a + 36b + c. \quad (4)$$

Now, solving these four-variable equations might be mind-boggling, but with restrictions $0 \leq a, b, c < 6$, we can more confidently get to work. In particular, we can restrict the possible values of the right-hand side of Equations 3 and 4 to the interval $[1296a, 1296a + 185]$ (where $185 = 5 \cdot 36 + 5$) for each possible value of a. We first do this for possible values of $7n^2 + 5n + 1$ (we leave solving the quadratic inequalities to the reader):

- When $a = 1$, we want n such that $1296 \leq 7n^2 + 5n + 1 \leq 1481$. The only positive integral solution for n is $n = 14$, in which case $7n^2 + 5n + 1 = 1443$. So, $36b + c = 1443 - 1296 = 147$, which has the solution $b = 4$ and $c = 3$. Thus, we compute $B = 7(14) + 2 = 100$ as one solution.
- When $a = 2$, we want n such that $2592 \leq 7n^2 + 5n + 1 \leq 2777$. The only positive integral solution is $n = 19$, in which case $7n^2 + 5n + 1 = 2623$. So, $36b + c = 2623 - 2592 = 31$, which yields no solutions (b, c) in base 6 digits.
- When $a = 3$, we want n such that $3888 \leq 7n^2 + 5n + 1 \leq 4073$. There are no positive integral values of n that satisfy these inequalities.
- When $a = 4$, we want n such that $5184 \leq 7n^2 + 5n + 1 \leq 5369$. The only positive integral solution for n is $n = 27$, in which case $7n^2 + 5n + 1 = 5239$. So, $36b + c = 5239 - 5184 = 55$, which yields no solutions (b, c) in base 6 digits.
- When $a = 5$, we want n such that $6480 \leq 7n^2 + 5n + 1 \leq 6665$. There are no positive integral values of n that satisfy these inequalities.

Now we check for possible values of $7n^2 + 9n + 3$:

- When $a = 1$, we want n such that $1296 \leq 7n^2 + 9n + 3 \leq 1481$. The only positive integral solution for n is $n = 13$, in which case $7n^2 + 9n + 3 = 1303$. So, $36b + c = 1303 - 1296 = 7$, which yields no solutions (b, c) in base 6 digits.
- When $a = 2$, we want n such that $2592 \leq 7n^2 + 9n + 3 \leq 2777$. The only positive integral solution is $n = 19$, in which case $7n^2 + 9n + 3 = 2701$. So, $36b + c = 2701 - 2592 = 109$, which yields $(b, c) = (3, 1)$. Thus, we compute $B = 7(19) + 4 = 137$ as a solution.
- When $a = 3$, we want n such that $3888 \leq 7n^2 + 9n + 3 \leq 4073$. The only positive integral solution for n is $n = 23$, in which case $7n^2 + 9n + 3 = 3913$. So, $36b + c = 3913 - 3888 = 25$, which yields no solutions (b, c) in base 6 digits.
- When $a = 4$, we want n such that $5184 \leq 7n^2 + 9n + 3 \leq 5369$. The only positive integral solution for n is $n = 27$, in which case $7n^2 + 9n + 3 = 5349$. So, $36b + c = 5349 - 5184 = 165$, which yields no solutions (b, c) in base 6 digits.
- When $a = 5$, we want n such that $6480 \leq 7n^2 + 9n + 3 \leq 6665$. The only positive integral solution for n is $n = 30$, in which case $7n^2 + 9n + 3 = 6573$. So, $36b + c = 6573 - 6480 = 93$, which yields no solutions (b, c) in base 6 digits.

The sum of the solutions for B is $100 + 137 = 237$.

Follow Up Problem: For which other bases λ are there solutions to $(111)_B = (aabbcc)_\lambda$?

46. Let (x, y, z) be an ordered triplet of real numbers that satisfies the following system of equations:

$$x + y^2 + z^4 = 0,$$
$$y + z^2 + x^4 = 0,$$
$$z + x^2 + y^4 = 0.$$

If m is the minimum possible value of $\lfloor x^3 + y^3 + z^3 \rfloor$, find the modulo 2007 residue of m.

Answer: 2006

Solution: By the Trivial Inequality,

$$\begin{array}{ll} x^2 \geq 0, & x^4 \geq 0, \\ y^2 \geq 0, & y^4 \geq 0, \\ z^2 \geq 0, & z^4 \geq 0. \end{array}$$

Thus, we have each of the following:

$$\begin{array}{lcl} y^2 + z^4 \geq 0 & \Leftrightarrow & x + y^2 + z^4 \geq x, \\ z^2 + x^4 \geq 0 & \Leftrightarrow & y + z^2 + x^4 \geq y, \\ x^2 + y^4 \geq 0 & \Leftrightarrow & z + x^2 + y^4 \geq z. \end{array}$$

These inequalities, along with the given system of equations, show that each x, y, and z must be nonpositive.

Now, due to the cyclic symmetry of the given equations, it suffices to examine only the cases $0 \geq x \geq y \geq z$ and $0 \geq x \geq z \geq y$:

- If $0 \geq x \geq y \geq z$, then corresponding terms in the left-hand side of the first equation are at least as large as those in the third. Subtracting the third given equation from the first, we get
$$(x - z) + (y^2 - x^2) + (z^4 - y^4) = 0. \tag{5}$$
But,
$$x - z \geq 0,$$
$$y^2 - x^2 \geq 0,$$
$$z^4 - y^4 \geq 0.$$
So, equality must hold in each of these inequalities. Since the variables are all nonpositive, we have that $x = y = z$.
- If $0 \geq x \geq z \geq y$, then we substract the second equation from the first to get
$$(x - y) + (y^2 - z^2) + (z^4 - x^4) = 0.$$
And as before, each pair of grouped terms on the left-hand side but be at least 0, so they are all 0, and thus $x = y = z$.

Since $x = y = z$, substitution into each of the equations gives us a single equation in one variable:

$$x + x^2 + x^4 = 0 \qquad \Leftrightarrow \qquad x(1 + x + x^3) = 0.$$

So, either $x = y = z = 0$, or else $x = y = z = \lambda$, where λ is the real zero of the monotonically increasing polynomial $g(x) = 1 + x + x^3$. Since $g(-1) < 0 < g(-2/3)$, we know that $-1 < \lambda < -2/3$.

Now, we note that $1 + \lambda + \lambda^3 = 0$, so $\lambda^3 = -1 - \lambda$, thus

$$-1/3 < -1 - \lambda < 0 \qquad \Leftrightarrow \qquad -1/3 < \lambda^3 < 0.$$

Hence,

$$\lfloor x^3 + y^3 + z^3 \rfloor = \lfloor 3\lambda^3 \rfloor = -1,$$

which is less than $\lfloor x^3 + y^3 + z^3 \rfloor$, when $x = y = z = 0$.

47. Let $\{X_n\}$ and $\{Y_n\}$ be sequences defined as follows:

$$X_0 = Y_0 = X_1 = Y_1 = 1,$$

$$\begin{aligned} X_{n+1} &= X_n + 2X_{n-1} \qquad (n = 1, 2, 3, \ldots), \\ Y_{n+1} &= 3Y_n + 4Y_{n-1} \qquad (n = 1, 2, 3, \ldots), \end{aligned}$$

Let k be the largest integer that satisfies all of the following conditions:

(i) $|X_i - k| \leq 2007$, for some positive integer i;

(ii) $|Y_j - k| \leq 2007$, for some positive integer j; and

(iii) $k < 10^{2007}$.

Find the remainder when k is divided by 2007.

Answer: 1447

Credit: The ideas behind this problem were inspired by 1973 USAMO #2. This has long been one of my favorite USAMO problems perhaps because it was one of the very first I ever solved, and certainly the first that inspired me to dig deeper into relationships not denoted in the official solution.

Solution: Before digging into the recursion, we first get a grip on the three given conditions for k. Rephrased in English, our given task is to find the largest k less than 10^{2007} that is within 2007 of a member of each sequence. This suggests that we look to maximize terms of each sequence that have a difference no greater than $2 \cdot 2007 = 4014$. As we will see, (iii) adds no additional restriction to the answer.

The given recursive equations have something in common, which is the relationship between coefficients on their right-hand sides. To make this similarity clear, note that we can add a term to both sides of each equation to create equations that highlight sums of consecutive terms:

$$\begin{aligned} X_{n+1} + X_n &= 2(X_n + X_{n-1}) \qquad (n = 1, 2, 3, \ldots), \\ Y_{n+1} + Y_n &= 4(Y_n + Y_{n-1}) \qquad (n = 1, 2, 3, \ldots). \end{aligned}$$

By making both sides of each equation look "the same," we have implicitly highlighted two new, much simpler recursions. Let

$$S_n = X_{n+1} + X_n \qquad (n = 1, 2, 3, \ldots),$$
$$T_n = Y_{n+1} + Y_n \qquad (n = 1, 2, 3, \ldots),$$

Then,

$$S_n = 2S_{n-1} \qquad (n = 1, 2, 3, \ldots),$$
$$T_n = 4T_{n-1} \qquad (n = 1, 2, 3, \ldots),$$

where $S_0 = T_0 = 2$.

Now, we quickly see that $S_n = 2^{n+1}$ and $T_n = 2^{2n+1}$ for $n = 0, 1, 2, \ldots$.

The differences between terms in the sequences grow at a rate that is virtually exponential, so we can simply compute enough terms to see where the differences between terms grow larger than 4014:

X :	3,	5,	11,	21,	43,	85,	171,	341,
	683,	1365,	2731,	5461,	10923,	21845,	43691,	...
Y :	7,	25,	103,	409,	1639,	6553,	26215,	104857,
	419431,	...						

The largest pair of terms, one in each sequence, that are within 4014 of one another are 5461 and 6553. The largest k which is within 2007 of them both is $5461 + 2007 = 7468$. The remainder when 7468 is divided by 2007 is 1447.

48. Let a and b be relatively prime positive integers such that a/b is the maximum possible value of

$$\sin^2 x_1 + \sin^2 x_2 + \sin^2 x_3 + \cdots + \sin^2 x_{2007},$$

where, for $1 \leq i \leq 2007$, x_i is a nonnegative real number, and

$$x_1 + x_2 + x_3 + \cdots + x_{2007} = \pi.$$

Find the value of $a + b$.

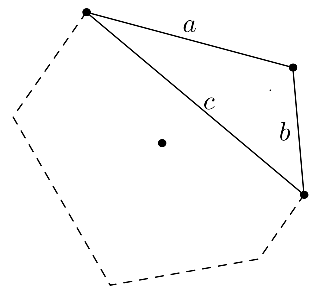
Answer: 13

Credit: This problem is adapted from one on the 1985 International Mathematical Olympiad shortlist.

Solution: Let S be the sum of the squares of the sines as in the problem. The sum of the angles is a nice number, π, which leads us to look for some kind of relationship to a circle. In particular, the sum of squares of sines we wish to maximize is modeled by the squares of the sides of a 2007-gon inscribed in a circle of diameter 1.

We proceed more generally with a lemma.

Lemma: Given an n-gon inscribed in a circle of diameter 1, $n \geq 4$, at least one of the interior angles is at least $\pi/2$. Dropping that vertex to create an (n-1)-gon does not reduce the value of S (which is now the sum of the squares of $n - 1$ sines).



Proof of lemma: Consider the non-acute triangle formed by the non-acute vertex and its neighbors. By the Law of Cosines, $c^2 \geq a^2 + b^2$. Dropping the non-acute vertex from the n-gon means replacing $a^2 + b^2$ with c^2, which results in a sum at least as large as before.

Continually dropping vertices leads us to a triangle, whose vertices we label A, B, and C, and so it is now our task to find the maximum value of

$$S' = AB^2 + BC^2 + CA^2.$$

Using vectors, we have

$$PA^2 + PB^2 + PC^2 = 3GP^2 + GA^2 + GB^2 + GC^2. \tag{6}$$

So, we pick some convenient and revealing points to be P. first, we let $P = O$, the circumcenter of the triangle. Thus Equation 6 includes the circumradius, R, whose value is known:

$$OA^2 + OB^2 + OC^2 = 3GO^2 + GA^2 + GB^2 + GC^2.$$

which is to say

$$3R^2 - 3GO^2 = GA^2 + GB^2 + GC^2. \tag{7}$$

Next, we let $P = A$, $P = B$, and $P = C$ in turn to get the system of equations

$$AB^2 + AC^2 = 4GA^2 + GB^2 + GC^2,$$
$$AB^2 + BC^2 = GA^2 + 4GB^2 + GC^2,$$
$$AC^2 + BC^2 = GA^2 + GB^2 + 4GC^2.$$

Adding these three new equations, and dividing the result by 2, we now have

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2). \tag{8}$$

Now, we can combine Equation 8 with 7 to get

$$AB^2 + BC^2 + CA^2 = 3(3R^2 - 3GO^2) = 9R^2 - 9GO^2.$$

Finally, we note that

$$9R^2 - 9GO^2 \geq 9R^2,$$

where equality occurs if $G = O$, which happens for an equilateral triangle. So, the maximum value of S' is $9R^2 = 9(1/2)^2 = 9/4$. Now, the maximum for S is at most $9/4$. We can achieve this maximum value of S by letting $x_4 = x_5 = x_6 = \cdots = x_{2007} = 0$. So, $a + b = 9 + 4 = 13$.

49. How many 7-element subsets of $\{1, 2, 3, \ldots, 14\}$ are there, the sum of whose elements is divisible by 14?

Answer: 246

Credit: This problem is a special case of the last problem from the 1995 IMO.

Solution: Let $U = \{1, 2, 3, \ldots, 14\}$. The sum of all 14 elements in U is $(14 \cdot 15)/2 = 105$, which is odd. Now, consider disjoint 7-element subsets A and B of U (such that $A \cup B = U$ and $A \cap B = \emptyset$). Since $7 \mid 105$, the sum of the elements of A is a multiple of 7 if and only if the sum of the elements of B is a multiple of 7. However, 105 is odd, so the sum of the elements of A has opposite parity of the sum of the elements of B. So, if both sums are multiples of 7, exactly one of them is a multiple of 14. This means we can count the number of subsets of U that are multiples of 7 and divide by 2 to get the total number which are multiples of 14.

Now, consider the r elements in a 7-element subset of U that are equal to 7 or less. For instance, if $A = \{2, 4, 8, 9, 10, 13, 14\}$, then we consider the elements 2 and 4. So long as $0 < r < 7$, we can create a related group of subsets of U by continually add 1 to the elements 2 and 4, and reducing modulo 7 when the elements grow larger than 7. For instance,

$$\begin{aligned}
A &= \{2, 4, 8, 9, 10, 13, 14\},\\
A_1 &= \{3, 5, 8, 9, 10, 13, 14\},\\
A_2 &= \{4, 6, 8, 9, 10, 13, 14\},\\
A_3 &= \{5, 7, 8, 9, 10, 13, 14\},\\
A_4 &= \{1, 6, 8, 9, 10, 13, 14\},\\
A_5 &= \{2, 7, 8, 9, 10, 13, 14\},\\
A_6 &= \{1, 3, 8, 9, 10, 13, 14\}.
\end{aligned}$$

Since $2r$ and 7 are relatively prime, these 7 subsets of U have sums which are distinct modulo 7. This means they make up the full range of modulo 7 residues. In other words, exactly one of them has elements that add up to a multiple of 7.

There are $\binom{14}{7} = 3432$ 7-elements subsets of U. The two subsets $\{1, 2, \ldots, 7\}$ and $\{8, 9, \ldots, 14\}$ have elements that sum to multiples of 7. Of the other 3430 such subsets, the cycling of modulo 7 residues above shows us that exactly $3430/7 = 490$ of them have elements that sum to multiples of 7. In total 492 of the 7-element subsets of U have elements that sum to a multiple of 7, of which $492/2 = 246$ have elements that sum to a multiple of 14.

50. A block Z is formed by gluing one face of a solid cube with side length 6 onto one of the circular faces of a right circular cylinder with radius 10 and height 3 so that the centers of the square and circle coincide. If V is the smallest convex region that contains Z, calculate $\lfloor \mathrm{vol}\, V \rfloor$ (the greatest integer less than or equal to the volume of V).

Answer: 1882

Credit: Problem and solution by Harvard University undergraduate Zachary Abel.

Solution: In figure 3, the cube has vertices $A_1A_2A_3A_4B_1B_2B_3B_4$, point O is the center of square $A_1A_2A_3A_4$ as well as the circle, and radii OC_i $(1 \le i \le 4)$ are drawn perpendicular to sides $A_{i-1}A_i$. (Note: the indices are understood cyclically, so $A_0 = A_4$, etc.)

Construct a region W as the union of the following pieces, as detailed in Figures 4 to 50: region α is simply the cylinder with radius $r = 10$ and height $h = 3$; regions β_i $(1 \le i \le 4)$

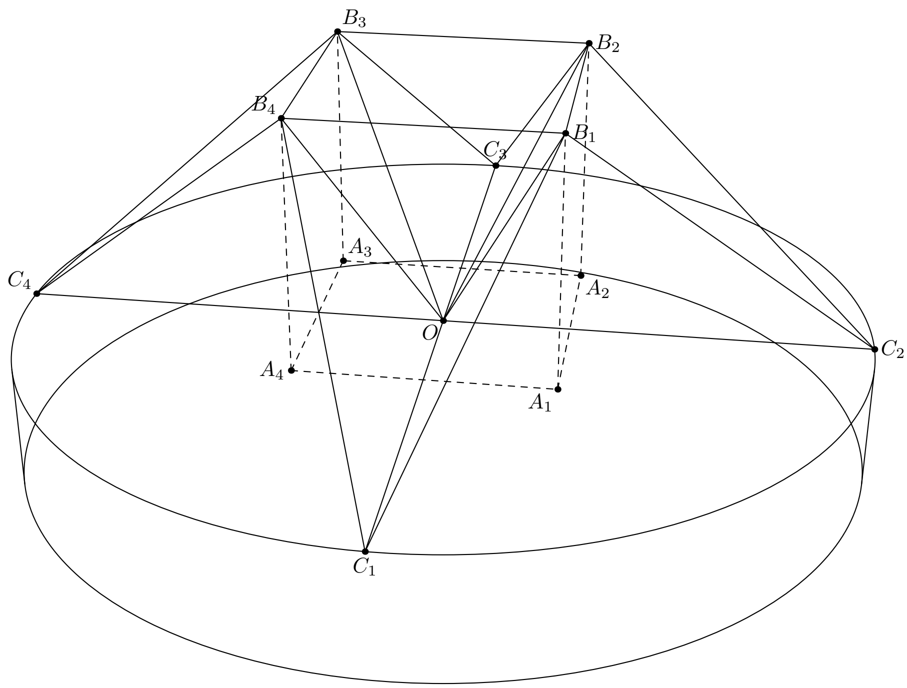


Figure 3: Block V

are cones whose bases are quarter circles formed by minor arc C_iC_{i+1} and with vertices at B_i; regions γ_i ($1 \le i \le 4$) are tetrahedra with vertices at C_i, B_{i-1}, B_i, and O; and finally, region δ is a pyramid with base $B_1B_2B_3B_4$ and vertex O. It is clear that W contains all of Z.

Notice that each piece composing W must be contained in V, by V's convexity. Indeed, each region β_i is simply the union of all line segments B_iX where X is in the quarter disk formed by arc C_iC_{i+1}, each piece γ_i is the union of all line segments XY with X on edge $B_{i-1}B_i$ and Y on segment C_iO, and regions α and δ are subsets of Z itself. Thus, W is a subset of V.

On the other hand, it can be verified that W itself is convex. This means V is contained in W, since V is the smallest convex region containing all of Z. Thus, $W = V$, so we must simply calculate the volume of W.

- (Region α): Region α is a cylinder with volume $\pi(10)^2(3) = 300\pi$.
- (Regions β_i): Region β_1 is a cone whose base has area $\frac{1}{4}\pi(10)^2 = 25\pi$ and whose height is $|B_1A_1| = 6$, so it has volume $\frac{1}{3}(25\pi)(6) = 50\pi$. Thus, the four βs contribute a volume of 200π.
- (Regions γ_i): Let D_1 be the midpoint of A_4A_1, and notice that tetrahedra $B_4D_1B_1O$ and $B_4D_1B_1C_1$ are right pyramids whose base $B_4D_1B_1$ has area $\frac{1}{2}(6)(6) = 18$ and which have heights of length $D_1O = 3$ and $D_1C_1 = 10 - 3 = 7$. So their total volume, i.e. the volume of γ_1, is $\frac{1}{3}(18)(3+7) = 60$, meaning the γs contribute a volume of 240.
- (Region δ): This is simply a pyramid with base of area $6^2 = 36$ and height of 6, so a volume of $\frac{1}{3}(36)(6) = 72$.

So $\operatorname{vol} W = \operatorname{vol} V = 300\pi + 200\pi + 240 + 72 = 312 + 500\pi$. Using $3.14 < \pi < 3.142$ gives $1882 < \operatorname{vol} V < 1883$, i.e. $\lfloor \operatorname{vol} V \rfloor = 1882$.

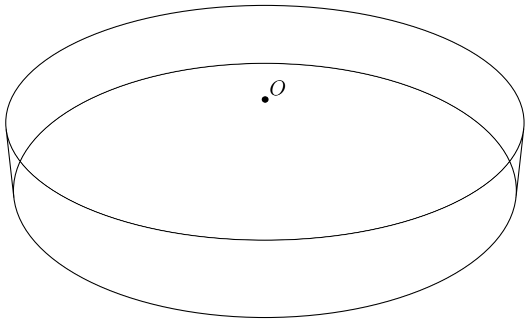


Figure 4: Region α

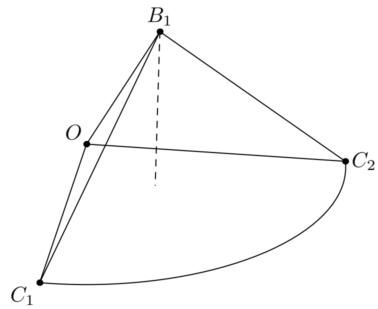


Figure 5: Region β_1

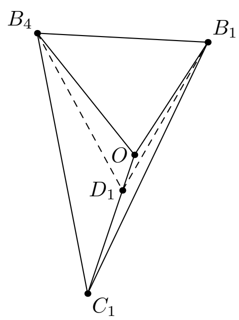


Figure 6: Region γ_1

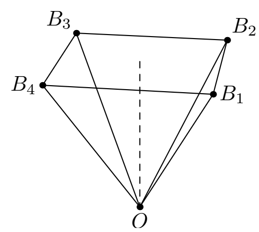


Figure 7: Region δ

51. Find the highest point (largest possible y-coordinate) on the parabola

$$y = -2x^2 + 28x + 418.$$

Answer: 516

Solution: We complete the square in order to express the function (right-hand side of the equation) in a form that is easier to examine:

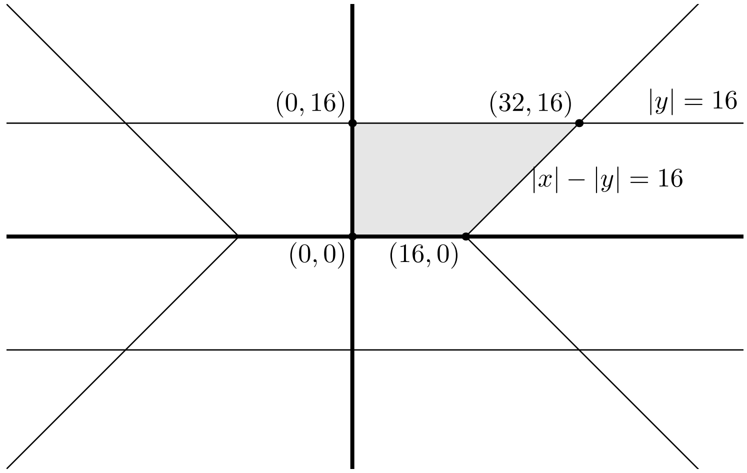
$$\begin{aligned} -2x^2 + 28x + 418 &= -2(x^2 - 14x) + 418 \\ &= -2\left(x^2 - 14x + \left(\frac{-14}{2}\right)^2\right) + 418 - (-2)\left(\frac{-14}{2}\right)^2 \\ &= -2(x^2 - 14x + 49) + 418 + 98 \\ &= -2(x-7)^2 + 516. \end{aligned}$$

By the Trivial Inequality, for real x, we have $(x-7)^2 \geq 0$, which equality if and only if $x = 7$. Thus,

$$-2(x-7)^2 \leq 0 \qquad \Leftrightarrow \qquad -2(x-7)^2 + 516 \leq 516,$$

which equality if and only if $x = 7$. That is to say that $(7, 516)$ is the vertex of the parabola, where 516 is the highest the parabola rises above the x-axis.

52. Let $T =$ TNFTPP. Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq T - 500$ and $|y| \leq T - 500$. Find the area of region R.



Answer: 1536

Solution: Since $T - 500 = 516 - 500 = 16$, we desire to compute the area of the region bounded by

$$\begin{aligned} |x| - |y| &\leq 16, \\ |y| &\leq 16. \end{aligned}$$

The second inequality involves only one variable, so it is the easier to deal with. It tells us that the y coordinate of any point in region R is at distance no more than 16 from the x-axis.

The first inequality is easier to deal with if we recognize the symmetry of its graph by quadrant:

$$|x| - |y| = |-x| - |y| = |x| - |-y| = |-x| - |-y|,$$

so (x, y) is in region R if and only if $(-x, y)$, $(x, -y)$, and $(-x, -y)$ are all in region R. This means we can find the area of R in Quadrant I, and multiply that area by 4 to find the total area of R.

In Quadrant I, the inequalities because $x - y \leq 16$ and $y \leq 16$, and the boundaries of these inequalities are the lines $x - y = 16$ and $y = 16$. Along with the axes, the part of region R in Quadrant I is a trapezoid with coordinates $(0, 0)$, $(0, 16)$, $(32, 16)$, $(16, 0)$. We can find its area in a number of ways, one of which is multiplying its height, 16, by the average of the bases, which have lengths 16 and 32:

$$16\left(\frac{16 + 32}{2}\right) = 16 \cdot 24 = 384.$$

So, the total area of region R is $4 \cdot 384 = 1536$.

53. Let T = TNFTPP. Three distinct positive Fibonacci numbers, all greater than T, are in arithmetic progression. Let N be the smallest possible value of their sum. Find the remainder when N is divided by 2007.

Answer: 501

Solution: Let F_a, F_b, and F_c be three Fibonacci numbers in arithmetic progression, where $F_a < F_b < F_c$. Note that this means that $a < b < c$ as well. Then $2F_b = F_a + F_c$, so $2F_b > F_c$. However,

$$2F_b < F_b + F_{b+1} = F_{b+2}.$$

Hence, $F_{b+2} > 2F_b > F_c$, so $b+2 > c$. But $c > b$, so it must be that $c = b+1$. In other words, if three distinct positive Fibonacci numbers are in arithmetic progression, then the two larger of them are consecutive Fibonacci numbers:

$$2F_b = F_a + F_c \qquad \Leftrightarrow \qquad 2F_b = F_a + F_{b+1}.$$

Isolating F_a and simplifying (using the recursive definition of Fibonacci numbers), we get

$$F_a = 2F_b - F_{b+1} = 2F_b - (F_{b-1} + F_b) = F_b - F_{b-1} = F_{b-2}.$$

Thus, the three Fibonacci numbers are F_{b-2}, F_b, and F_{b+1} for any positive integer b.

Now, we want all three of the Fibonacci numbers to be greater than 1536, so we compute some Fibonacci numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, \ldots.$$

The arithmetic progression we are looking for is 1597, 4181, 6765, and their sum is

$$1597 + 4181 + 6765 = 3 \cdot 4181 = 12543.$$

So,

$$12543 = 6 \cdot 2007 + 501,$$

where 501 is the remainder.

54. Let $T = \text{TNFTPP}$. Consider the sequence $(1, 2007)$. Inserting the difference between 1 and 2007 between them, we get the sequence $(1, 2006, 2007)$. Repeating the process of inserting differences between numbers, we get the sequence $(1, 2005, 2006, 1, 2007)$. A third iteration of this process results in $(1, 2004, 2005, 1, 2006, 2005, 1, 2006, 2007)$. A total of 2007 iterations produces a sequence with $2^{2007} + 1$ terms. If the integer $4T$ (that is, 4 times the integer T) appears a total of N times among these $2^{2007} + 1$ terms, find the remainder when N gets divided by 2007.

Answer: 1589

Solution: The length of this solution betrays the fact that many solvers probably found the answer by playing around, without complete knowledge of some facets of the sequences involved. Hopefully, some solvers learned a lot from exploring this problem, and perhaps a bit more from reading this solution.

Before getting started, it's worth noting that $T = 501$, so our primary goal is to determine how to compute the number of instances of 2004 in the final sequence. Some students might have noticed that the problem would be considerably more or less challenging if our goal were to compute the total number of instances of some other integers.

Let's take a look at each of two possible paths of exploration:

Exploration 1: It might take a while, and a couple of pieces of paper, but we can simply count the number of instances of each integer in the sequences that result from the first few iterations:

Iterations	$2007's$	$2006's$	$2005's$	$2004's$	$2003's$	$2002's$	$2001's$	$2000's$	$1999's$
1	1	1							
2	1	1	1						
3	1	2	2	1					
4	1	2	4	3	1				
5	1	3	6	7	4	1			
6	1	3	9	13	11	5	1		
7	1	4	12	22	24	16	6	1	
8	1	4	16	34	46	40	22	7	1

Some very talented pattern hunters might note that, for all the cells to the right of the 2006 column, each cell is the sum of the numbers in the cell directly above it and the cell to its upper-left. For instance, the number of instances of 2004 on the sequence produced by the 8^{th} iteration is the sum of the number of instances of 2003 and the number of instances of 2004 in the sequence produced by the 7^{th} iteration.

Talented pattern hunters might then recognize that the counts in the column below 2005 alternate between the perfect squares and products of consecutive integers. Problem solvers might jump at this point to computations, though in order to be most sure, we explore these patterns further.

Exploration 2: Some students might get to this point after some work similar to that above. Other students, particularly those with some experience with recursive sequences, might start at this point. Here we examine the process that produces each sequence. Each sequence begins like the following, and continues this way until the very end, which varies slightly:

$$(1, a_1, a_2, 1, a_3, a_4, 1, a_5, a_6, 1, a_7, a_8, 1, \ldots.$$

Most importantly, the process of taking differences ensures that for each integer $k \geq 1$,

$$|a_k - a_{k+1}| = 1.$$

The next iteration would look like

$$(1, a_1 - 1, a_1, 1, a_2, a_2 - 1, 1, a_3 - 1, a_3, 1, a_4, a_4 - 1, 1, \ldots.$$

So, we notice several things happen upon each iteration:

(i) Each previous term remains.

(ii) A new instance of 2006 occurs if and only if the number of the iteration is odd.

(iii) For integers $2 \leq k \leq 2005$, a new instance of k occurs for each previous instance of $k+1$.

Now we compute the 2007th number in the 2004's column, which is the sum of the first 2006 elements in the 2005's column:

$$\begin{aligned}
\sum_{n=1}^{1003} n^2 + \sum_{n=1}^{1003} n(n-1) &= \sum_{n=1}^{1003} n^2 + \sum_{n=1}^{1003} n^2 - n \\
&= 2\sum_{n=1}^{1003} n^2 - \sum_{n=1}^{1003} n \\
&= 2\left(\frac{1003 \cdot 1004 \cdot 2007}{6}\right) - \frac{1003 \cdot 1004}{2} \\
&= 1003 \cdot 1004 \cdot 669 - 1003 \cdot 502.
\end{aligned}$$

Our advantage at this point is that we don't need to completely tally the total. We just need the remainder when we divide by 2007. Since 669 is exactly a third of 2007, and 1003 is just under half of 2007, we "compute around" these numbers. First, we note that

$$1003 \cdot 1004 \equiv 1 \cdot 2 \equiv 2 \pmod{3}.$$

Now we find the remainder without significant computation:

$$\begin{aligned}
1003 \cdot 1004 \cdot 669 - 1003 \cdot 502 &\equiv 2 \cdot 669 - 2006 \cdot 251 \\
&\equiv 1338 - (-1) \cdot 251 \\
&\equiv 1338 + 251 \equiv 1589 \pmod{2007}.
\end{aligned}$$

Note: In the first step of the computations above, the fact that $2007/669 = 3$ allows us to reduce the product $1003 \cdot 1004$ modulo 3, which is congruent to 2.

55. Let $T =$ TNFTPP, and let $R = T - 914$. Let x be the smallest real solution of

$$3x^2 + Rx + R = 90x\sqrt{x+1}.$$

Find the value of $\lfloor x \rfloor$.

Answer: 225

Solution: The equation is $3x^2 + 675x + 675 = 90x\sqrt{x+1}$, the ugliest part of which is the radical. Bravely, we rid ourselves of the radical through the substitution $y = \sqrt{x+1}$. It helps to note that $x = y^2 - 1$:

$$3y^4 + 669y^2 + 3 = 90(y^2 - 1)y.$$

Expanding the right-hand-side and grouping the terms on one side, we find a polynomial with coefficients whose magnitudes are symmetric about the middle term:

$$3y^4 - 90y^3 + 669y^2 + 90y + 3 = 0.$$

Now, since $y \neq 0$, we can divide both sides of the equation by y^2 without fear of losing any roots:

$$3y^2 - 90y + 669 + 90y^{-1} + 3y^{-2} = 0. \tag{9}$$

Equation 9 allows us to apply an interesting tactic. Since the substitution $y = -1/y$ leaves the equation unchanged, we can take advantage of the symmetry of coefficient magnitudes using the substitution $k = y - 1/y$:

$$\begin{aligned} 3y^2 - 90y + 669 + 90y^{-1} + 3y^{-2} &= 3y^2 - 6 + 3y^{-2} - 90(y - y^{-1}) + 675 \\ &= 3(y - y^{-1})^2 - 90(y - y^{-1}) + 675 \\ &= 3k^2 - 90k + 675 = 3(k - 15)^2 = 0. \end{aligned}$$

The only possible value of k is $k = 15$, so we have

$$y - y^{-1} = 15 \qquad \Rightarrow \qquad y^2 - 15y - 1 = 0.$$

Solving, we get $y = \frac{15 \pm \sqrt{229}}{2}$. However, since $y = \sqrt{x+1}$, we have that $y \geq 0$, so $y = \frac{15+\sqrt{229}}{2}$, and thus

$$x = y^2 - 1 = \left(\frac{15 + \sqrt{229}}{2}\right)^2 - 1 = \frac{225 + 30\sqrt{229} + 229}{4} - 1 = \frac{225 + 15\sqrt{229}}{2}.$$

All that remains is to determine the integer part of x. Note that we could use the Binomial Expansion Theorem for radicals:

$$\sqrt{229} = \left(15^2 + 2^2\right)^{\frac{1}{2}},$$

but we have a more clever approximation method in mind. Since

$$15\sqrt{229} = \sqrt{225} \cdot \sqrt{229} = \sqrt{227 - 2} \cdot \sqrt{227 + 2} = \sqrt{227^2 - 2^2} < 227,$$

we know that

$$225 + 15\sqrt{229} < 225 + 227 = 452.$$

Also, $15 < \sqrt{229}$, so

$$450 < 225 + 15\sqrt{229} < 452 \qquad \Leftrightarrow \qquad 225 < x < 226,$$

hence $\lfloor x \rfloor = 225$.

56. Let $T = \text{TNFTPP}$. In the binary expansion of

$$\frac{2^{2007} - 1}{2^T - 1},$$

how many of the first 10,000 digits to the right of the radix point are 0's?

Answer: 810

Solution: One way of viewing repeating decimals that is often enlightening, is as infinite geometric series. For instance, in decimal notation, we have

$$\begin{aligned} 0.1\overline{36} &= \frac{1}{10} + \frac{36}{10^3} + \frac{36}{10^5} + \frac{36}{10^7} + \cdots \\ &= \frac{1}{10} + \frac{\frac{36}{10^3}}{1 - \frac{1}{100}} \\ &= \frac{1}{10} + \frac{36}{990} = \frac{135}{990} = \frac{3}{22}. \end{aligned}$$

Applying the same idea to the given fraction in reverse, we see that

$$\frac{n}{2^{225}-1} = \frac{\frac{n}{2^{225}}}{1-\frac{1}{2^{225}}} = \frac{n}{2^{225}} + \frac{n}{(2^{225})^2} + \frac{n}{(2^{225})^3} + \cdots$$

where n is the remainder when $2^{2007}-1$ is divided by $2^{225}-1$.

We can divide $2^{2007}-1$ by $2^{225}-1$ a bit more easily with a substitution such as $x = 2^9$, and then employing a method such as synthetic division. One execution of the division is as follows:

$$\begin{aligned}
\frac{x^{223}-1}{x^{25}-1} &= x^{23}\cdot\frac{x^{200}-1}{x^{25}-1} + \frac{x^{23}-1}{x^{25}-1} \\
&= x^{23}\left(\frac{(x^{25})^8-1}{x^{25}-1}\right) + \frac{x^{23}-1}{x^{25}-1} \\
&= x^{23}[(x^{25})^7 + (x^{25})^6 + (x^{25})^5 + (x^{25})^4 + (x^{25})^3 + (x^{25})^2 + x^{25} + 1] + \frac{x^{23}-1}{x^{25}-1} \\
&= x^{198} + x^{173} + x^{148} + \cdots + x^{23} + \frac{x^{23}-1}{x^{25}-1},
\end{aligned}$$

where the fractional part, after substitution, is equal to

$$\frac{2^{207}-1}{2^{225}-1}.$$

Now, using $n = 2^{207}-1$ in the geometric series we derived before, we have

$$\frac{2^{207}-1}{2^{225}-1} = \frac{2^{207}-1}{2^{225}} + \frac{2^{207}-1}{(2^{225})^2} + \frac{2^{207}-1}{(2^{225})^3} + \cdots,$$

in which each repeating block of 225 binary digits after the radix point displays the binary representation of $2^{207}-1$. Since the binary representation of 2^{207} is a 1 followed by 207 0's, the binary representation of $2^{207}-1$ is the next smaller binary number, which is simply the binary number consisting of only 207 consecutive 1's.

So, the repeating block of the radix expansion consists of 225 digits, of which the final 207 are all 1's (and the previous 18 are 0's). Since $10000/225 = 44$ R 100, the first 10000 digits to the right of the radix point include 44 full cycles of the repeating block, but the first 100 digits of the block. The 44 cycles include $44 \cdot 18$ total 0's, with an extra 18 0's in the front of the next 100 digits, for a total of $45 \cdot 18 = 810$ 0's in the first 10000 digits to the right of the radix point.

57. Let $T =$ TNFTPP. How many positive integers are within T of exactly $\lfloor\sqrt{T}\rfloor$ perfect squares? (Note: $0^2 = 0$ is considered a perfect square.)

Answer: 57

Credit: This problem was adapted from one on the 1994 Putnam Exam.

Solution: We begin by writing the given condition mathematically, as inequalities. We are looking for integers that are within 810 of exactly 28 perfect squares. Let N be such an integer, then

$$(m+27)^2 \;\le\; N+810 \;\le\; (m+28)^2-1,$$

$$m^2 \;\ge\; N-810 \;\ge\; (m-1)^2+1,$$

where $m^2, (m+1)^2, \ldots, (m+27)^2$ are the 28 perfect squares. Subtracting our second inequality from our first quickly yields a chain of inequalities of a single, linear variable:

$$54m+729 \le 1620 \le 58m+781,$$

from which we immediately determine that either $m=15$ or $m=16$.

- If $m=15$, then plugging into our first two inequality chains, we get

$$1764 \;\le\; N+810 \;\le\; 1848 \qquad \Leftrightarrow \qquad 954 \;\le\; N \;\le\; 1038,$$

$$225 \;\ge\; N-810 \;\ge\; 197 \qquad \Leftrightarrow \qquad 1035 \;\ge\; N \;\ge\; 1007.$$

  Together, we have $1007 \le N \le 1035$, which yields $1035-1007+1=29$ integers N that fit the problem description.

- If $m=16$, then the same two inequality chains become

$$1849 \;\le\; N+810 \;\le\; 1935 \qquad \Leftrightarrow \qquad 1039 \;\le\; N \;\le\; 1125,$$

$$256 \;\ge\; N-810 \;\ge\; 226 \qquad \Leftrightarrow \qquad 1066 \;\ge\; N \;\ge\; 1036.$$

  Together, we have $1039 \le N \le 1066$, which yields $1066-1039+1=28$ integers N that fit the description.

In total, there are $29+28=57$ positive integers that are within 810 of exactly 28 perfect squares.

58. Let $T=$ TNFTPP. For natural numbers $k, n \ge 2$, we define $S(k,n)$ such that

$$S(k,n) = \left\lfloor \frac{2^{n+1}+1}{2^{n-1}+1} \right\rfloor + \left\lfloor \frac{3^{n+1}+1}{3^{n-1}+1} \right\rfloor + \cdots + \left\lfloor \frac{k^{n+1}+1}{k^{n-1}+1} \right\rfloor.$$

Compute the value of $S(10, T+55) - S(10,55) + S(10, T-55)$.

Answer: 339

Credit: The function in this problem appeared on the 1992 Austrian Mathematical Olympiad.

Solution: We take a look at any given piece of the general function:

$$\left\lfloor \frac{m^{n+1}+1}{m^{n-1}+1} \right\rfloor.$$

The numerator of the fraction is approximately m^2 times the denominator, so we note that

$$\frac{m^{n+1}+1}{m^{n-1}+1} = m^2 - \frac{m^2-1}{m^{n-1}+1}.$$

We note two cases:

- When $n \geq 3$, we note that

$$m^2 - 1 \leq m^{n-1}+1 \qquad \Rightarrow \qquad \frac{m^2-1}{m^{n-1}+1} \leq 1,$$

  in which case

$$m^2 - 1 \leq m^2 - \frac{m^2-1}{m^{n-1}+1} < m^2 \qquad \Rightarrow \qquad \left\lfloor \frac{m^{n+1}+1}{m^{n-1}+1} \right\rfloor = m^2 - 1.$$

- When $n = 2$, we have

$$\frac{m^{n+1}+1}{m^{n-1}+1} = \frac{m^3+1}{m+1} = m^2 - m + 1.$$

In the first case,

$$S(k,n) = \sum_{m=2}^{k} (m^2 - 1),$$

for all values of $n \geq 3$. Thus, $S(10,112) = S(10,57)$, hence

$$S(10,112) - S(10,57) + S(10,2) = S(10,2).$$

Now we compute the remaining expression from the second case:

$$\begin{aligned} S(10,2) &= \sum_{m=2}^{10} (m^2 - m + 1) = \sum_{m=1}^{10} (m^2 - m + 1) - 1 \\ &= \sum_{m=1}^{10} m^2 - \sum_{m=1}^{10} m + \sum_{m=1}^{10} 1 - 1 \\ &= \frac{10 \cdot 11 \cdot 21}{6} - \frac{10 \cdot 11}{2} + 10 - 1 \\ &= 385 - 55 + 10 - 1 = 339. \end{aligned}$$

59. Let T = TNFTPP. Fermi and Feynman play the game *Probabicloneme* in which Fermi wins with probability a/b, where a and b are relatively prime positive integers such that $a/b < 1/2$. The rest of the time Feynman wins (there are no ties or incomplete games). It takes a negligible amount of time for the two geniuses to play *Probabicloneme*, so they play many many times. Assuming they can play infinitely many games (eh, they're in Physicist Heaven, we can bend the rules), the probability that they are ever tied in total wins after

they start (they have the same positive win totals) is $(T - 322)/(2T - 601)$. Find the value of a.

Answer: 17

Solution: Many students probably tried equating some complicated infinite sums to solve this problem. If there is such a solution, I have not yet found it.

The probability that the physicists are ever tied is $17/77$. Let $p = a/b$, and let $q = 1 - p$, so that q is the probability that Feynman defeats Fermi during any one game. The key observation is that there is a symmetry in the set of outcomes in which the players end up tied. For some positive integer k, there is a one-to-one correspondence between the following categories of results:

(i) Fermi wins the very first game, but the players each win k of the first $2k$ games.

(ii) Feynman wins the very first game, but the players each win k of the first $2k$ games.

We establish this one-to-one correspondence by noting that changing the outcomes of all $2k$ games turns a set of outcomes (i) into a set of outcomes (ii), and vice versa.

Now, we'd like to put our observation to work. We let m be the probability that, from any given point, Fermi is ever ahead of Feynman (just one game will do) in wins. Similarly, we let n be the probability that, from any given point, Feynman ever wins at least one more game than Fermi. So, we can summarize the symmetry we noticed with the equation

$$mq = np,$$

and in particular,

$$mq + np = \frac{17}{77}.$$

We'll need more than this lonely equation, so we start to think about relationships involving m or n. We note that we can break m into two parts: the probability that Fermi wins the very next game, p, and the probability that he loses the next game, q, times the square of the probability that Fermi wins one more game than he loses from any particular point in time (because now he's go to do that not just once, but twice), m. So,

$$m = p + qm^2. \tag{10}$$

Similarly,

$$n = q + pn^2. \tag{11}$$

Now that we have several equations in four variables, we can take a deep breath and move forward. Fortunately, one of our equations, namely $q = 1 - p$, is plainly begging to be used as a substitution equation. We turn Equations 10 and 11 into two-variable equations:

$$m = p + (1 - p)m^2,$$
$$n = (1 - p) + pn^2.$$

Rearranging, we can view the first of these equations as a quadratic in m, and the second as a quadratic in n:

$$(1 - p)m^2 - m + p = 0,$$
$$pn^2 - n + (1 - p) = 0.$$

Noting that $p < 1/2$, we apply the Quadratic Formula on the first quadratic to get

$$m = \frac{1 \pm \sqrt{1 - 4(1-p)p}}{2(1-p)} = \frac{1 \pm (1-2p)}{2(1-p)},$$

so either $m = p/q$, or $m = 1$. From the second quadratic, we have

$$n = \frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p} = \frac{1 \pm (1-2p)}{2p},$$

so either $n = 1$, or $n = q/p$.

Since $q > p$, we have $q/p > 1$, so $n \neq q/p$. Since $n = 1$ and $mq = np = p$, we see that $m = p/q$. So,

$$\frac{17}{77} = mq + np = 2np = 2p \qquad \Rightarrow \qquad p = \frac{17}{154}.$$

Thus, $a = 17$.

Note: an intuitive argument suggests that since Feynman is the better player, he will win one more game that Fermi after any point based on limits or the law of large numbers. In other words, $n = 1$. This means that $mq = np \Leftrightarrow mq = p$, so $m = p/q$, which helps us avoid so much algebra.

60. Let $T = \text{TNFTPP}$. Triangle ABC has $AB = 6T - 3$ and $AC = 7T + 1$. Point D is on BC so that AD bisects angle BAC. The circle through A, B, and D has center O_1 and intersects line AC again at B', and likewise the circle through A, C, and D has center O_2 and intersects line AB again at C'. If the four points B', C', O_1, and O_2 lie on a circle, find the length of BC.

Answer: 111

Credit: Problem and solution by Harvard University undergraduate Zachary Abel.

Solution: Let $\triangle ABC$ have angles $\alpha = \angle A$, $\beta = \angle B$, and $\gamma = \angle C$, and let O and O_3 be the circumcenters of triangles ABC and $AB'C'$ respectively. Circles O and O_1 have chord AB in common, so $OO_1 \perp AB$. Likewise, so $O_2O_3 \perp AC'$, i.e. O_2O_3 is parallel to OO_1. We similarly find $OO_2 \parallel O_1O_3$, so $OO_1O_3O_2$ is a parallelogram. Finally, circles O_1 and O_2 have a common chord AD, so $O_1O_2 \perp AD$, which means $\angle OO_1O_2 = \angle BAD = \frac{\alpha}{2}$. Likewise $\angle OO_2O_1 = \frac{\alpha}{2}$, implying $OO_1O_3O_2$ is in fact a rhombus.

We can calculate $\angle AO_1O_2 = \frac{1}{2}\angle AO_1D = \frac{1}{2}(2\beta) = \beta$ and likewise $\angle AO_2O_1 = \gamma$, so $ABC \sim AO_1O_2$ and $\angle O_1AO_2 = \alpha$. And since $\angle O_1OO_2 = 180^\circ - \frac{\alpha}{2} - \frac{\alpha}{2} = 180^\circ - \alpha$, quadrilateral AO_1OO_2 is cyclic.

We have $O_3O_1 = O_3O_2$ on the rhombus, and $O_3B' = O_3C'$, so O_3 lies on the perpendicular bisectors of both O_1O_2 and $B'C'$. Assuming these two lines do not coincide, O_3 must be the center of the given circle ω through O_1, O_2, B', and C'. (The exceptional case in fact does not occur, but the details of this proof have been postponed until after the rest of the solution.) This means that ω is circle O_3 itself, meaning $A \in \omega$. And since AO_1OO_2 is cyclic, O also lies on ω. Finally, since $O_3O_1 = O_3O = O_3O_2 = O_1O = OO_2$, the rhombus has angles of 60° and 120°. Thus, $\angle BAC = \angle O_1AO_2 = \frac{1}{2}\angle O_1O_3O_2 = 60^\circ$, so

$$BC = \sqrt{AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos 60^\circ} = 111.$$

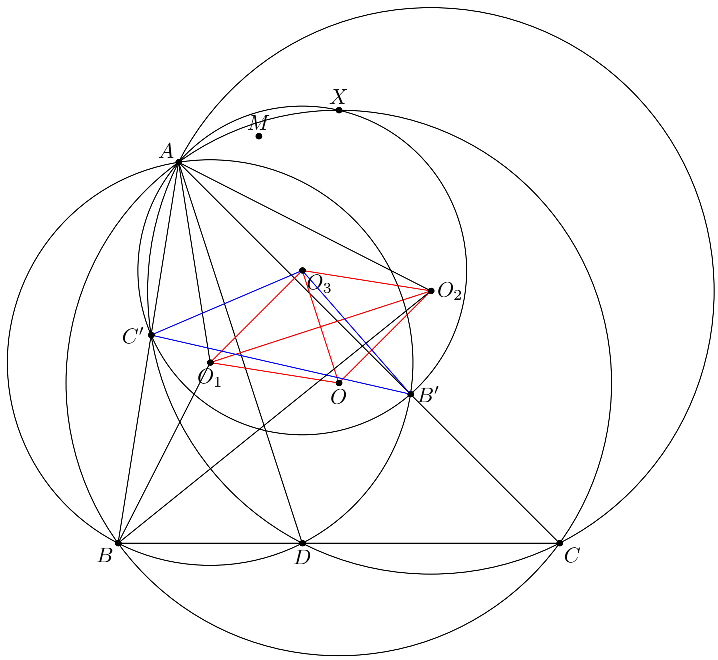


Figure 8: Problem 60

It remains to see that the perpendicular bisectors of O_1O_2 (namely OO_3) and of $B'C'$ never coincide. To that end, first note that $OO_3 \perp O_1O_2 \perp AD$, i.e. $OO_3 \parallel AD$. Thus, circles O and O_3 pass through another common point X which has $XA \perp OO_3 \parallel AD$. This puts X on the diameter orthogonal to BC (since AX is the external angle bisector at A), so $XB = XC$, i.e. $X \neq A$. And since OO_3 passes through M, the midpoint of AX, it cannot pass through X itself. But by similar reasoning on circle O_3, X is on the perpendicular bisector of $B'C'$, i.e. this line cannot coincide with OO_3, as needed.

TB1. The sum of the digits of an integer is equal to the sum of the digits of three times that integer. Prove that the integer is a multiple of 9.

Solution: Let $S(n)$ be the sum of the digits of an integer n. Now we can write the given information as an equation: $S(n) = S(3n)$. It is also true that an integer is congruent to the sum of its digit modulo 9 (try to prove this fact if you have not before). Thus,

$$S(n) \equiv n \pmod{9},$$
$$S(3n) \equiv 3n \pmod{9}.$$

Now, subtracting the first equation from the second, we get

$$S(3n) - S(n) \equiv 3n - n \pmod{9}.$$

Since $S(n) = S(3n)$, we have that $S(3n) - S(n) = 0$, and so

$$S(3n) - S(n) = 0 \equiv 2n \pmod{9} \qquad \Rightarrow \qquad 0 \equiv n \pmod{9},$$

which is to say that n is a multiple of 9.

Follow Up Problem: For which integers k is it always true that

$$S(n) = S(kn) \qquad \Rightarrow \qquad n \equiv 0 \pmod{9}?$$

TB2. Factor completely over integer coefficients the polynomial $p(x) = x^8 + x^5 + x^4 + x^3 + x + 1$. Demonstrate that your factorization is complete.

Motivation: The given polynomial is closely related to a polynomial that is much easier to factor:

$$\begin{aligned} f(x) = x^5 + x^4 + x^3 + x^2 + x + 1 &= \frac{x^6 - 1}{x - 1} = \frac{(x^3 + 1)(x^3 - 1)}{x - 1} \\ &= \frac{(x + 1)(x^2 - x + 1)(x - 1)(x^2 + x + 1)}{x - 1} \\ &= (x + 1)(x^2 - x + 1)(x^2 + x + 1). \end{aligned}$$

Due to the common ratio between terms, we were able to rewrite the polynomial f using the formula for a finite geometric series. From there, we factored using differences of squares, then differences of cubes. We can factor one of the quadratic factors further into linear factors with integer coefficients only if the roots are rational, which they aren't in either case, so we are done.

The polynomial p looks a lot like f, but does not have the same geometric series structure. There are at least a couple of ideas that could help us proceed.

Idea (a): We can try multiplying p by $x - 1$, hoping to exploit the similarities between p and f:

$$(x - 1)(x^8 + x^5 + x^4 + x^3 + x + 1) = x^9 - x^8 + x^6 - x^3 + x^2 - 1.$$

The alternating signs help us spot two similar chunks of this polynomial, which help us factor it completely:

$$\begin{aligned} x^9 - x^8 + x^6 - x^3 + x^2 - 1 &= x^6(x^3 - x^2 + 1) - (x^3 - x^2 + 1) \\ &= (x^3 - x^2 + 1)(x^6 - 1) \\ &= (x^3 - x^2 + 1)(x^3 + 1)(x^3 - 1) \\ &= (x^3 - x^2 + 1)(x + 1)(x - 1)(x^2 + x + 1)(x^2 - x + 1). \end{aligned}$$

Idea (b): The roots of f are the sixth roots of unity besides 1. We note that

$$p(x) - f(x) = x^8 - x^2 = x^2(x^6 - 1),$$

which has roots $x = 0$, along with the sixth roots of unity. The five roots in common between f and $p - f$ must also be roots of $f + (p - f) = p$. The fifth degree polynomial with these roots is just f, which is to say that f is a factor of p. Now we note that

$$\frac{p(x)}{f(x)} = x^3 - x^2 + 1.$$

Solution: We claim that

$$(x + 1)(x^2 + x + 1)(x^2 - x + 1)(x^3 - x^2 + 1) = x^8 + x^5 + x^4 + x^3 + x + 1$$

is the complete factorization of p into polynomials with integer coefficients.

The polynomial multiplication is mathematically trivial, if a mild computational chore, and we already know that the linear factor and quadratic factors cannot be factored further over integral coefficients. If we factor $x^3 - x^2 + 1$ further, one of the factors must be linear. If that factor has integral coefficients, then its root is rational. This rational root must therefore be a root of $x^3 - x^2 + 1$. But by the Rational Roots Theorem, the only possible rational roots of $x^3 - x^2 + 1$ are ± 1, neither of which turns out to be a root of $x^3 - x^2 + 1$. So, our factorization above is a complete factorization of p into factors with integral coefficients.

TB3. 4014 boys and 4014 girls stand in a line holding hands, such that only the two people at the ends are not holding hands with exactly two people (an ordinary line of people). One of the two people at the ends gets tired of the hand-holding fest and leaves. Then, from the remaining line, one of the two people at the ends leaves. Then another from an end, and then another, and another. This continues until exactly half of the people from the original line remain. Prove that no matter what order the original 8028 people were standing in, that it is possible that exactly 2007 of the remaining people are girls.

Motivation: Finding another way to phrase a problem (in a more plain mathematical way) is often helpful. Since $8028/4 = 2007$, our goal is to prove that in the beginning, there is some "run" of 4014 consecutive people such that half are boys and half are girls. That way, people can leave from the ends until only that run of 4014 boys and girls remains.

Now, if exactly half the people remaining are girls, then exactly half the people who left are also girls. This means that we can model the problem as if the people are equally spaced in a circle. We want to show that we can draw a diameter that seperates the group into two halves, each of which contain an equal number of boys and girls.

Solution: We draw a diameter, L_1, that divides the group around the circle into halves. We label one half the "cool people" and the other half the "others". Let $G_c(1)$ be the number of girls who are cool people, and let $G_o(1)$ be the number of girls who are others. Regardless of position, we have that $G_c(1) + G_o(1) = 4014$.

Now, consider a slight clockwise rotation of the diameter such that exactly one cool person becomes an other, and exactly one other becomes a cool person. This new diameter is L_2. Likewise, we can continue rotations all the way around the circle until we have diameters $L_1, L_2, L_3, \ldots, L_{8028}$. And, for each n, we have that

$$G_c(n) + G_o(n) = 4014.$$

Note also that obviously $|G_c(n) - G_c(n+1)| \leq 1$, and when G_c changes, the changes are discrete.

Either $G_c(1) = 2007$, in which case we are done, or else $G_c \neq 2007$. If $G_c \neq 2007$, then we note that L_{2008} is the same diameter as L_1, but with the cool people and others swapped entirely. Thus, $G_c(2008) = 4014 - G_c(1) = G_o(1)$. Since 2007 lies in between $G_c(1)$ and $G_o(1)$, which are on opposite sides of 2007 on the number line, we know by discrete continuity that there is some n, $1 < n < 2008$, for which $G_c(n) = 2007$, and we are done.

TB4. Circle O is the circumcircle of non-isosceles triangle ABC. The tangent lines to circle O at points B and C intersect at L_a, and the tangents at A and C intersect at L_b. The external angle bisectors of triangle ABC at B and C intersect at I_a, and the external bisectors at A and C intersect at I_b. Prove that lines L_aI_a, L_bI_b, and AB are concurrent.

Credit: Problem and solution by Harvard University undergraduate Zachary Abel.

Solution: Define L_c and I_c analagously, and notice that I_a, I_b, and I_c are the excenters of triangle ABC. As the internal angle bisector at A, namely I_aA, and the external angle bisector, I_bI_c, are perpendicular, it follows that I_aA, I_bB, and I_cC are the altitudes of $\triangle I_aI_bI_c$, i.e. ABC is the orthic triangle of $I_aI_bI_c$. Hence, circle O is the nine point circle of $\triangle I_aI_bI_c$. So if we define M_a as the midpoint of I_bI_c, and likewise for M_b and M_c, these three points also lie on circle O.

Define $P_a = BC \cap M_bM_c$, and likewise for P_b and P_c. We will show that I_a, L_a, and P_c are collinear. The proof that I_b, L_b, and P_c are collinear will be analagous, and then it will follow that lines I_aL_a, I_bL_b, and AB concur at P_c, as desired. To show that I_a, L_a, and P_c are collinear, we verify in turn that the line $P_bP_c = z$ contains the points I_a, $X = BM_c \cap CM_b$, and finally L_b.

First we show $I_a \in z$. By Pascal's theorem on hexagon $ABM_bM_aM_cC$, points $AB \cap M_aM_c = R$, $BM_b \cap M_cC = I_a$, and $M_bM_a \cap AC = S$ are collinear. Let AM_a intersect this line at T, and set $U = M_bM_c \cap M_aI_a$, which is the midpoint of M_bM_c since $M_bM_aM_cI_a$ is a parallelogram.

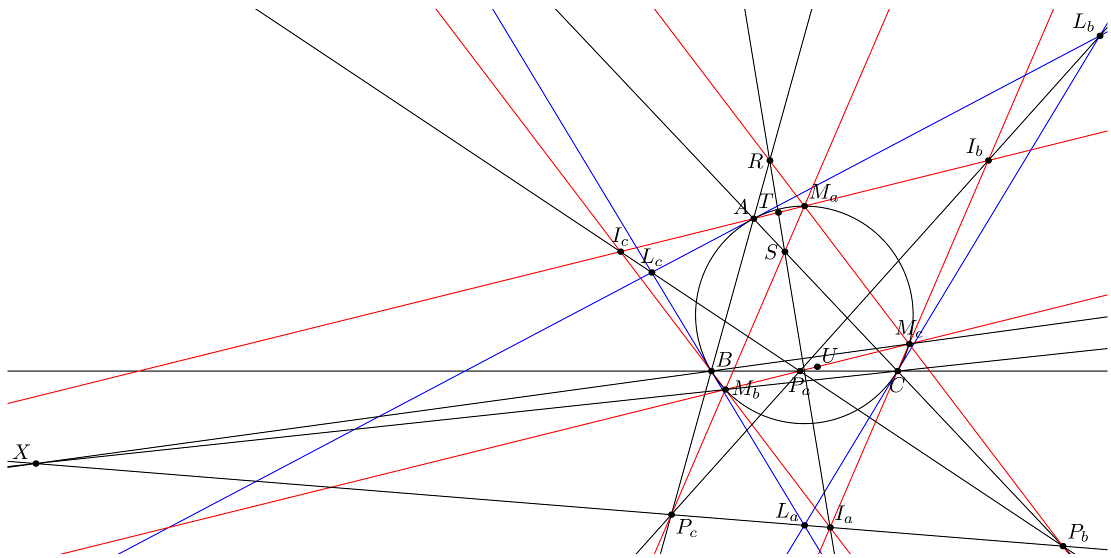


Figure 9: Tie Breaker 4

Using Cr as the cross ratio operator, the perspectivity at M_a tells us that

$$\begin{aligned}&\mathrm{Cr}(R,S;T,I_a)\\&=\mathrm{Cr}(M_aR,M_aS;M_aT,M_aI_a)\\&=\mathrm{Cr}(M_aM_c,M_aM_b;M_a\infty,M_aU)\\&=\mathrm{Cr}(M_c,M_b;\infty,U)\\&=-1,\end{aligned}$$

where ∞ is the point at infinity along line M_aA (or equivalently, along line M_bM_c) in the projective plane. But by Lemma below applied to quadrilateral $RASM_a$, the line through $RA\cap SM_a = P_c$ and $RM_a \cap AS = P_c$ intersects line RS at the harmonic conjugate of T with respect to RS, i.e. it intersects at I_a. So I_a is indeed on line z.

Next, by Pascal's theorem on hexagon $ABM_cM_aM_bC$, points $AB\cap M_aM_b = P_c$, $BM_c\cap M_bC = X$, and $M_cM_a\cap CA = P_b$ are collinear, i.e. $X\in z$.

Finally, by (a degenerate case of) Pascal's theorem on hexagon BBM_bCCM_c, points $BB\cap CC = L_a$ (by convention we use the tangent line in this case), $BM_b\cap CM_c = I_a$, and $M_bC\cap M_cB = X$ are collinear, i.e. L_a is on line $I_aX = z = P_bP_c$. This concludes the proof.

lemma: For any four points A,B,C,D not all collinear, define $E = AC\cap BD$, $F = AB\cap CD$, $G = AD\cap BC$, and $H = FG\cap AC$. Then $A,C;E,H$ form a harmonic set, i.e. $\mathrm{Cr}(A,C;E,H) = 1$, where Cr denotes the cross ratio operator.

proof: Without loss of generality, we may assume that none of the labelled points is at infinity. By Ceva's theorem on triangle ACD with cevians through B, we have

$$\frac{AE}{EC}\cdot\frac{CF}{FD}\cdot\frac{DG}{GA} = 1.$$

Likewise, by Menelaus's theorem on the same triangle with transversal HFG, we have

$$\frac{AH}{HC}\cdot\frac{CF}{FD}\cdot\frac{DG}{GA} = -1.$$

Combining these two equations gives

$$\mathrm{Cr}(A, C; E, H) = \frac{AE/EC}{AH/HC} = -1,$$

as needed.