

The 2008 iTest Tournament of Champions Guide

The 2008 iTest TOC Round 1 Problems

1. (2 points) Find k where 2^k is the largest power of 2 that divides the product

$$2008 \cdot 2009 \cdot 2010 \cdots 4014.$$

2. (3 points) Let

$$\begin{aligned} A &= 5 \cdot 6 - 6 \cdot 7 + 7 \cdot 8 - \cdots + 2003 \cdot 2004, \\ B &= 1 \cdot 10 - 2 \cdot 11 + 3 \cdot 12 - \cdots + 1999 \cdot 2008. \end{aligned}$$

Find the value of $A - B$.

3. (5 points) Simon and Garfunkle play in a round-robin golf tournament. Each player is awarded one point for a victory, a half point for a tie, and no points for a loss. Simon beat Garfunkle in the first game by a record margin as Garfunkle sent a shot over the bridge and into troubled waters on the final hole. Garfunkle went on to score 8 total victories, but no ties at all. Meanwhile, Simon wound up with exactly 8 points, including the point for a victory over Garfunkle. Amazingly, every other player at the tournament scored exactly n . Find the sum of all possible values of n .
4. (7 points) The rules for the movement of a king on a chessboard are as follows: The king can legally move to any of the (up to 8) squares adjacent diagonally or on a side. Andrew places a king on an ordinary 8×8 chessboard. He then makes 64 total moves with the king such that the king visits every square on the board, never crosses its own path, and winds up at its original position (where Andrew first placed it). Along the way, Andrew counts the number of times the king moves diagonally (from one square to another that shares no side). Call that number M . Find the maximum possible value of M .
5. (8 points) Let $c_1, c_2, c_3, \dots, c_{2008}$ be complex numbers such that

$$|c_1| = |c_2| = |c_3| = \cdots = |c_{2008}| = 1492,$$

and let $S(2008, t)$ be the sum of all products of these 2008 complex numbers taken t at a time. Let Q be the maximum possible value of

$$\left| \frac{S(2008, 1492)}{S(2008, 516)} \right|.$$

Find the remainder when Q is divided by 2008.

The 2008 iTest TOC Round 2 Problems

1. (3 points) Find the smallest positive integer n such that there are at least three distinct ordered pairs (x, y) of positive integers such that

$$x^2 - y^2 = n.$$

2. (4 points) Find the value of $|xy|$ given that x and y are integers and

$$6x^2y^2 + 5x^2 - 18y^2 = 17253.$$

3. (5 points) A regular 2008-gon is located in the Cartesian plane such that $(x_1, y_1) = (p, 0)$ and $(x_{1005}, y_{1005}) = (p + 2, 0)$, where p is prime and the vertices,

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_{2008}, y_{2008}),$$

are arranged in counterclockwise order. Let

$$S = (x_1 + y_1i)(x_3 + y_3i)(x_5 + y_5i) \cdots (x_{2007} + y_{2007}i),$$

$$T = (y_2 + x_2i)(y_4 + x_4i)(y_6 + x_6i) \cdots (y_{2008} + x_{2008}i).$$

Find the minimum possible value of $|S - T|$.

4. (6 points) Find the maximum of $x + y$ given that x and y are positive real numbers that satisfy

$$x^3 + y^3 + (x + y)^3 + 36xy = 3456.$$

5. (7 points) While running from an unrealistically rendered zombie, Willy Smithers runs into a vacant lot in the shape of a square, 100 meters on a side. Call the four corners of the lot corners 1, 2, 3, and 4, in clockwise order. For $k = 1, 2, 3, 4$, let d_k be the distance between Willy and corner k . Let

(a) $d_1 < d_2 < d_4 < d_3$,

(b) d_2 is the arithmetic mean of d_1 and d_3 , and

(c) d_4 is the geometric mean of d_2 and d_3 .

If d_1^2 can be written in the form $\frac{a - b\sqrt{c}}{d}$, where a, b, c , and d are positive integers, c is square-free, and the greatest common divisor of a, b , and d is 1, find the remainder when $a + b + c + d$ is divided by 2008.

The 2008 iTest TOC Round 3 Problems

- (3 points) Find the remainder when $712!$ is divided by 719.
- (4 points) Note that there are exactly three ways to write the integer 4 as a sum of positive odd integers where the order of the summands matters:

$$1 + 1 + 1 + 1 = 4,$$

$$1 + 3 = 4,$$

$$3 + 1 = 4.$$

Let $f(n)$ be the number of ways to write a natural number n as a sum of positive odd integers where the order of the summands matters. Find the remainder when $f(2008)$ is divided by 100.

- (5 points) Arthur stands on a circle drawn with chalk in a parking lot. It is sunrise and there are birds in the trees nearby. He stands on one of five triangular nodes that are spaced equally around the circle, wondering if and when the aliens will pick him up and carry him to their homeworld. Initially, Arthur faces the clockwise direction (from an overhead view) from the node he is standing on. He flips a fair coin 12 times, each time chanting the name of a nearby star system. Each time he flips a head, he walks around the circle, in the direction he is facing, until he reaches the next node in that direction. Each time he flips a tail, he reverses direction, then walks around the circle until he reaches the next node in that new direction. After 12 flips, Arthur finds himself on the node at which he started. He thinks this is fate, but Arthur is quite mistaken. If a and b are relatively prime positive integers such that a/b is the probability that Arthur flipped exactly 6 heads, find $a + b$.
- (6 points) Euclid places a morsel of food at the point $(0, 0)$ and an ant at the point $(1, 2)$. Every second, the ant walks one unit in one of the four coordinate directions. However, whenever the ant moves to $(x, \pm 3)$, Euclid's malicious brother Möbius picks it up and puts it at $(-x, \mp 2)$, and whenever it moves to $(\pm 2, y)$, his cousin Klein puts it at $(\mp 1, y)$. If p and q are relatively prime positive integers such that $\frac{p}{q}$ is the expected number of steps the ant takes before reaching the food, find $p + q$.
- (7 points) It is well-known that the n^{th} triangular number can be given by the formula $n(n + 1)/2$. A Pythagorean triple of *square numbers* is an ordered triple (a, b, c) such that $a^2 + b^2 = c^2$. Let a Pythagorean triple of *triangular numbers* (a PTTN) be an ordered triple of positive integers (a, b, c) such that $a \leq b < c$ and

$$\frac{a(a + 1)}{2} + \frac{b(b + 1)}{2} = \frac{c(c + 1)}{2}.$$

For instance, $(3, 5, 6)$ is a PTTN ($6 + 15 = 21$). Here we call both a and b *legs* of the PTTN. Find the smallest natural number n such that n is a leg of *at least* six distinct PTTNs.

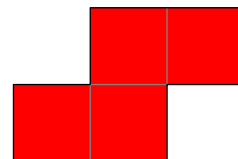
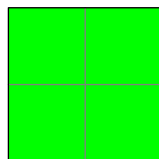
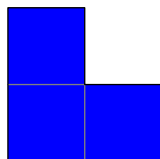
The 2008 iTest TOC Round 4 Problems

- (2 points) Yatta and Yogi play a game in which they begin with a pile of n stones. The players take turns removing 1, 2, 3, 5, 6, 7, or 8 stones from the pile. That is, when it is a player's turn to remove stones, that player may remove from 1 to 8 stones, but *cannot* remove exactly 4 stones. The player who removes the last stone *loses*. Yogi goes first and finds that he has a winning position, meaning that so long as he plays perfectly, Yatta cannot defeat him. For how many positive integers n from 100 to 2008 inclusive is this the case?
- (4 points) Let A the number of 12-digit words that can be formed from the alphabet $\{0, 1, 2, 3, 4, 5, 6\}$ if each pair of neighboring digits must differ by exactly 1. Find the remainder when A is divided by 2008.
- (5 points) For how many integers $1 \leq n \leq 9999$ is there a solution to the congruence

$$\phi(n) \equiv 2 \pmod{12},$$

where $\phi(n)$ is the Euler phi-function?

- (6 points) Each of the 24 students in Mr. Friedman's class cut up a 7×7 grid of squares while he read them short stories by Mark Twain. While not all of the students cut their squares up in the same way, each of them cut their 7×7 square up into at most the three following types (shapes) of pieces



Let a , b , and c be the number of total pieces of each type from left to right respectively after all 24 7×7 squares are cut up. How many ordered triples (a, b, c) are possible?

- (8 points) For positive integers $m, n \geq 3$, let $h(m, n)$ be the maximum (finite) number of intersection points between a simple m -gon and a simple n -gon. (Note: a polygon is *simple* if it does not intersect itself.) Evaluate

$$\sum_{m=3}^6 \sum_{n=3}^{12} h(m, n).$$

The 2008 iTest TOC Round 5 Problems

1. (3 points) Let a , b , c , and d be positive real numbers such that $abcd = 17$. Let m be the minimum possible value of

$$a^2 + b^2 + c^2 + a(b + c + d) + b(c + d) + cd.$$

Compute $\lfloor 17m \rfloor$.

2. (4 points) Let N be the smallest natural number that, when written to its left, forms an integer with twice as many digits that is a perfect square. Find the remainder when N is divided by 1000.
3. (5 points) The 360 volumes of the *Encyclopedia Galactica* are out of order in the library. Fortunately for the librarian, the books are numbered. Due to his religion, which holds both encyclopedias and the concept of parity in high esteem, the librarian can only rearrange the books two at a time, and then only by switching the position of an even numbered volume with that of an odd numbered volume. Find the minimum number of such transpositions sufficient to get the books back into ordinary sequential order, regardless of the starting positions of the books. (Find the minimum number of transpositions in the worst-case scenario.)
4. (6 points) Let

$$f(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{1}{n-k} \binom{n-k}{k},$$

for each positive integer n . If $|f(2007) + f(2008)| = a/b$ for relatively prime positive integers a and b , find the remainder when a is divided by 1000.

5. (7 points) Two squares of side length 2 are glued together along their boundary so that the four vertices of the first square are glued to the midpoints of the four sides of the other square, and vice versa. This gluing results in a convex polyhedron. If the square of the volume of this polyhedron is written in simplest form as $\frac{a+b\sqrt{c}}{d}$, what is the value of $a + b + c + d$?

The 2008 iTest TOC Round 6 Problems

1. (2 points) Let

$$X = \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} + \cdots + \cos \frac{2006\pi}{7} + \cos \frac{2008\pi}{7}.$$

Compute $\lfloor 2008X \rfloor$.

2. (4 points) Jon wrote the n smallest perfect squares on one sheet of paper, and the n smallest triangular numbers on another (note that 0 is both square and triangular). Jon notices that there are the same number of triangular numbers on the first paper as there are squares on the second paper, but if n had been one smaller, this would not have been true. If $n < 2008$, let m be the greatest number Jon could have written on either paper. Find the remainder when m is divided by 2008.

3. (5 points) Let $\phi = \frac{1+\sqrt{5}}{2}$ be the positive root of $x^2 = x + 1$. Define a function $f : \mathbb{N} \rightarrow \mathbb{N}$ by

$$f(0) = 1$$

$$f(2x) = \lfloor \phi f(x) \rfloor$$

$$f(2x+1) = f(2x) + f(x).$$

Find the remainder when $f(2007)$ is divided by 2008.

4. (6 points) If m is a positive integer, let S_m be the set of rational numbers in reduced form with denominator at most m . Let $f(m)$ be the sum of the numerator and denominator of the element of S_m closest to e (Euler's constant). Given that $f(2007) = 3722$, find the remainder when $f(1000)$ is divided by 2008.

5. (8 points) Three circles with centers V_0, V_1, V_2 and radii 33, 30, 25 respectively are mutually externally tangent: P_i is the tangency point between circles V_{i+1} and V_{i+2} , where indices are taken modulo 3. For $i = 0, 1, 2$, line $P_{i+1}P_{i+2}$ intersects circle V_{i+1} at P_{i+2} and Q_i , and the same line intersects circle V_{i+2} at P_{i+1} and R_i . If Q_0R_1 intersects Q_2R_0 at X , then the distance from X to line R_1Q_2 can be expressed as $\frac{a\sqrt{b}}{c}$, where the integer b is not divisible by the square of any prime, and positive integers a and c are relatively prime. Find the value of $b + c$.

The 2008 iTest TOC Round 1 Solutions

1. Find k where 2^k is the largest power of 2 that divides the product

$$2008 \cdot 2009 \cdot 2010 \cdots 4014.$$

Answer: 2007

Solution: The product of consecutive integers reminds us a lot of factorials, encouraging us to rewrite the product as a ratio of factorials:

$$2008 \cdot 2009 \cdot 2010 \cdots 4014 = \frac{4014!}{2007!}.$$

Let 2^a be the greatest power of 2 that divides $4014!$, and let 2^b be the greatest power of 2 that divides $2007!$. Then, by Legendre's Theorem,

$$a = \left\lfloor \frac{4014}{2^1} \right\rfloor + \left\lfloor \frac{4014}{2^2} \right\rfloor + \left\lfloor \frac{4014}{2^3} \right\rfloor + \cdots,$$

$$b = \left\lfloor \frac{2007}{2^1} \right\rfloor + \left\lfloor \frac{2007}{2^2} \right\rfloor + \left\lfloor \frac{2007}{2^3} \right\rfloor + \cdots.$$

We could set out to compute both a and b , but that would be a chore. Noting that $4014/2 = 2007$, we see that

$$\begin{aligned} a - b &= \left(\left\lfloor \frac{4014}{2^1} \right\rfloor + \left\lfloor \frac{4014}{2^2} \right\rfloor + \left\lfloor \frac{4014}{2^3} \right\rfloor + \cdots \right) - \left(\left\lfloor \frac{2007}{2^1} \right\rfloor + \left\lfloor \frac{2007}{2^2} \right\rfloor + \left\lfloor \frac{2007}{2^3} \right\rfloor + \cdots \right) \\ &= \left(\left\lfloor \frac{2007}{2^0} \right\rfloor + \left\lfloor \frac{2007}{2^1} \right\rfloor + \left\lfloor \frac{2007}{2^2} \right\rfloor + \cdots \right) - \left(\left\lfloor \frac{2007}{2^1} \right\rfloor + \left\lfloor \frac{2007}{2^2} \right\rfloor + \left\lfloor \frac{2007}{2^3} \right\rfloor + \cdots \right) \\ &= \left\lfloor \frac{2007}{2^0} \right\rfloor = 2007. \end{aligned}$$

All terms cancel except the first term of a , giving us the number of remaining powers of 2 represented in the given product.

2. Let

$$\begin{aligned} A &= 5 \cdot 6 - 6 \cdot 7 + 7 \cdot 8 - \cdots + 2003 \cdot 2004, \\ B &= 1 \cdot 10 - 2 \cdot 11 + 3 \cdot 12 - \cdots + 1999 \cdot 2008. \end{aligned}$$

Find the value of $A - B$.

Answer: 20

Solution: Interestingly, there are many impressive ways to approach this problem. Here is just one. Let $a_n = (n+4)(n+5)$ and $b_n = n(n+9)$, so that

$$\begin{aligned} A &= a_1 - a_2 + a_3 - \cdots + a_{1999}, \\ B &= b_1 - b_2 + b_3 - \cdots + b_{1999}. \end{aligned}$$

Now, we note that for all n ,

$$a_n - b_n = (n + 4)(n + 5) - n(n + 9) = 20.$$

So,

$$\begin{aligned} A - B &= (a_1 - b_1) - (a_2 - b_2) + (a_3 - b_3) - \cdots + (a_{1999} - b_{1999}) \\ &= \sum_{k=1}^{1999} (-1)^{k+1} (a_k - b_k) \\ &= \sum_{k=1}^{1999} (-1)^{k+1} \cdot 20 \\ &= 20 - 20 + 20 - \cdots + 20 = 20. \end{aligned}$$

3. Simon and Garfunkle play in a round-robin golf tournament. Each player is awarded one point for a victory, a half point for a tie, and no points for a loss. Simon beat Garfunkle in the first game by a record margin as Garfunkle sent a shot over the bridge and into troubled waters on the final hole. Garfunkle went on to score 8 total victories, but no ties at all. Meanwhile, Simon wound up with exactly 8 points, including the point for a victory over Garfunkle. Amazingly, every other player at the tournament scored exactly n . Find the sum of all possible values of n .

Answer: 29

Credit: This problem was inspired by one from the 1976 Canadian Mathematical Olympiad.

Solution: Let $m + 2$ be the number of other players in the tournament. The total number of points awarded is equal to the total number of games played. We can compute this total in two different ways, giving us an equation. The first way counts the total number of games in terms of the number of players, $m + 2$. The second way notes that the m players besides Simon and Garfunkle scored mn total points:

$$\binom{m+2}{2} = mn + 8 + 8 \quad \Leftrightarrow \quad \frac{(m+2)(m+1)}{2} = mn + 16.$$

We can isolate n to learn more about it:

$$m^2 + 3m + 2 = 2mn + 32 \quad \Leftrightarrow \quad 2mn = m^2 + 3m - 30 \quad \Leftrightarrow \quad n = \frac{m+3}{2} - \frac{15}{m}.$$

Since n can only obtain integer values and values that are half more than integer values, either $15/m$ is an integer or half an integer. Since Garfunkle scored 8 victories, we know that $m \geq 8$ as well. If $15/m$ is an integer, then $m = 15$. If $15/m$ is half an integer, then m is an even divisor of 30, so $m = 10$ or $m = 30$. Now we note that

- When $m = 10$, $n = \frac{10+3}{2} - \frac{15}{10} = 6.5 - 1.5 = 5$,
- When $m = 15$, $n = \frac{15+3}{2} - \frac{15}{15} = 9 - 1 = 8$, and

- When $m = 30$, $n = \frac{30+3}{2} - \frac{15}{30} = 16.5 - .5 = 16$.

These solutions all check out, so the sum of the possible values of n is $5 + 8 + 16 = 29$.

4. The rules for the movement of a king on a chessboard are as follows: The king can legally move to any of the (up to 8) squares adjacent diagonally or on a side. Andrew places a king on an ordinary 8×8 chessboard. He then makes 64 total moves with the king such that the king visits every square on the board, never crosses its own path, and winds up at its original position (where Andrew first placed it). Along the way, Andrew counts the number of times the king moves diagonally (from one square to another that shares no side). Call that number M . Find the maximum possible value of M .

Answer: 36

Credit: This problem is a harder version of an old chess puzzle. If you know the origin, please send it to crawford@mistacademy.com.

Solution: Consider the 28-move “king tour” around the border squares of the board. This tour involves 28 vertical and horizontal (non-diagonal) moves. We claim that this is the minimum number of non-diagonal moves (making $M = 64 - 28 = 36$). To demonstrate this claim, note that since the tour cannot cross its own path, the border squares must be visited in the same order as they are in the 28-move “border tour” (either in clockwise or counterclockwise order).

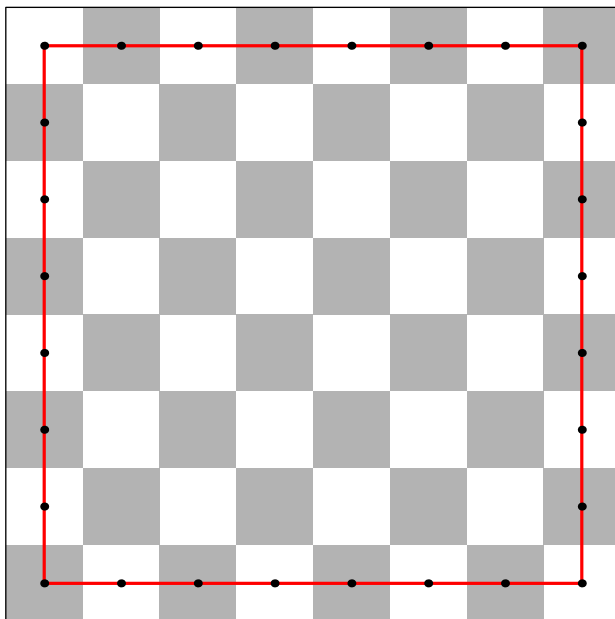


Figure 1: Border Tour

Now, label the squares (x, y) , where x, y are integers and $1 \leq x, y \leq 8$. Note that for a diagonal move, the parity of $x + y$ remains constant, but for a vertical or horizontal move, parity changes. When the king tour moves away from the 28-square border, it must make *at least* one vertical or horizontal move before returning to the border because parity must

5. Let $c_1, c_2, c_3, \dots, c_{2008}$ be complex numbers such that

$$|c_1| = |c_2| = |c_3| = \dots = |c_{2008}| = 1492,$$

and let $S(2008, t)$ be the sum of all products of these 2008 complex numbers taken t at a time. Let Q be the maximum possible value of

$$\left| \frac{S(2008, 1492)}{S(2008, 516)} \right|.$$

Find the remainder when Q is divided by 2008.

Answer: 448

Solution: Let $c_k = 1492z_k$ for $1 \leq k \leq 2008$ such that $z_k = \text{cis } \theta_k$ for some angle θ_k . Noting that $1492 + 516 = 2008$, we recognize that there are $\binom{2008}{1492} = \binom{2008}{516}$ terms in both the numerator and denominator. Letting $p_1, p_2, \dots, p_{\binom{2008}{1492}}$ be the products of the z_k taken 1492 at a time, and

$$P = z_1 z_2 \cdots z_{2008},$$

we have

$$\begin{aligned} \left| \frac{S(2008, 1492)}{S(2008, 516)} \right| &= \left| \frac{1492^{1492}(p_1 + p_2 + \cdots + p_{\binom{2008}{1492}})}{1492^{516} P \left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{\binom{2008}{1492}}} \right)} \right| \\ &= \left| \frac{1492^{976}(p_1 + p_2 + \cdots + p_{\binom{2008}{1492}})}{P \left(\overline{p_1} + \overline{p_2} + \cdots + \overline{p_{\binom{2008}{1492}}} \right)} \right| \\ &= 1492^{976} \cdot \left| \frac{1}{P} \right| \cdot \left| \frac{(p_1 + p_2 + \cdots + p_{\binom{2008}{1492}})}{(p_1 + p_2 + \cdots + p_{\binom{2008}{1492}})} \right| \\ &= 1492^{976} \cdot 1 \cdot 1 = 1492^{976}. \end{aligned}$$

Now we must compute the remainder when 1492^{976} is divided by 2008. Note:

$$2008 = 2^3 \cdot 251^1,$$

where 251 is prime. We know that $1492^{976} \equiv 0 \pmod{8}$, so must determine the remainder when 1492^{976} is divided by 251. By Fermat's Little Theorem,

$$1492^{250} \equiv (-14)^{250} \equiv (14)^{250} \equiv 1 \pmod{251},$$

so

$$1492^{976} \equiv (-14)^{976} \equiv (-14)^{226} \equiv (14)^{226} \pmod{251}.$$

Unfortunately, the next is a grind, though we can find clever ways to simplify as much of the computation as possible.

$$14^3 \equiv -17 \pmod{251},$$

$$14^4 \equiv 13 \pmod{251},$$

$$14^{10} \equiv -8 \pmod{251},$$

$$14^{22} \equiv -6 \pmod{251},$$

$$14^{45} \equiv 2 \pmod{251}.$$

So,

$$14^{226} \equiv (14^{45})^5 \cdot 14 \equiv 2^5 \cdot 14 \equiv 448 \pmod{251}.$$

Since $448 \equiv 0 \pmod{8}$, we have that $1492^{976} \equiv 448 \pmod{2008}$, so 448 is our answer.

The 2008 iTest TOC Round 2 Solutions

1. Find the smallest positive integer n such that there are at least three distinct ordered pairs (x, y) of positive integers such that

$$x^2 - y^2 = n.$$

Answer: 45

Solution: Factoring the left-hand side of the given equation allows us to compare factorizations of n to the sum and difference of x and y :

$$(x + y)(x - y) = n.$$

At this point, it is important that we note a restriction on the kinds of values of $x + y$ and $x - y$. Since $(x + y) - (x - y) = 2y \equiv 0 \pmod{2}$, the factors on the left-hand side necessarily have equal parity. This is in fact a necessary and sufficient condition as we can always find integer solutions to the system

$$\begin{aligned} x + y &= 2a \ (+1), \\ x - y &= 2b \ (+1), \end{aligned}$$

where a and b are integers. Students solving a problem like this might discover this restriction only after solving for x and y for various values of n .

Now, we are looking for either

- (a) integers n with at least 3 pairs of (6 total) even divisors $x + y > x - y$, or
- (b) integers n with at least 3 pairs of (6 total) odd divisors $x + y > x - y$.

Let p_1, p_2, \dots be distinct primes other than 2. In (a), we look for n in one of several forms:

- $2^7 = 128$,
- $2^2 \cdot p_1^2 \cdot p_2$, the smallest of which is $2^2 \cdot 3^2 \cdot 5^1 = 180$,
- $2^2 \cdot p_1^5$, the smallest of which is $2^2 \cdot 3^5 = 972$,
- $2^3 \cdot p_1^2$, the smallest of which is $2^3 \cdot 3^2 = 72$, or
- $2^4 \cdot p_1^1$, the smallest of which is $2^4 \cdot 3^1 = 48$,

[Proofreader Zachary Abel notes that these five cases can be combined into just two cases. Do you see why?] In (b), we look for n in one of the forms

- $p_1^2 \cdot p_2$, the smallest of which is $3^2 \cdot 5^1 = 45$,
- p_1^5 , the smallest of which is $3^5 = 243$.

Of all cases, the smallest possible n is 45.

2. Find the value of $|xy|$ given that x and y are integers and

$$6x^2y^2 + 5x^2 - 18y^2 = 17253.$$

Answer: 54

Solution: It's often good to look for a nice way to factor a complex expression, but the given equation is a bit intimidating. Letting $x^2 = m$ and $y^2 = n$ removes a bit of intimidation from the given equation, which becomes $6mn + 5m - 18n = 17253$. The left-hand side does not factor, but it looks like the product of two linear factors less (or more) a constant. Since 5 and 6 are relatively prime, there is only one good (nice) factoring option:

$$(m + _)(6n + _) = 6mn + 5m - 18n + \text{constant}.$$

The first blank must be a -3 so that we have $(-3)(6n) = -18n$. The second blank must be 5 so that we have $(m)(5) = 5m$. Thus, we have

$$(m - 3)(6n + 5) = 6mn + 5m - 18n - 15.$$

Now we can subtract 15 from both sides of the given equation and factor:

$$(x^2 - 3)(6y^2 + 5) = 17238.$$

Our variables are restricted to integer values, so it makes sense to compare the factorization of the left-hand side with a prime factorization of the right-hand side:

$$(x^2 - 3)(6y^2 + 5) = 2^1 \cdot 3^1 \cdot 13^2 \cdot 17^1.$$

It might be tough to pin down each of the factors on the left-hand side, except we know that the second is positive and congruent to 5 (mod 6). This means it does not include 2 or 3 among its factors. Also, since $13 \equiv 1 \pmod{6}$, it must include some other factor, which must be 17 in particular. Thus, one of the following must be true:

$$\begin{array}{lll} x^2 - 3 = 6 & \text{and} & 6y^2 + 5 = 2873, \\ x^2 - 3 = 78 & \text{and} & 6y^2 + 5 = 221, \\ x^2 - 3 = 1014 & \text{and} & 6y^2 + 5 = 17. \end{array}$$

We rule out the first set of equations by noting that

$$6y^2 = 2868 \quad \Rightarrow \quad y^2 = 478,$$

which has no solutions in integers. We rule out the last set of equations by noting that

$$6y^2 = 12 \quad \Rightarrow \quad y^2 = 2,$$

which has no solutions in integers.

The second set of equations does yield solutions in integers:

$$x^2 - 3 = 78 \quad \Rightarrow \quad x^2 = 81 \quad \Rightarrow \quad x = \pm 9,$$

and

$$6y^2 + 5 = 221 \quad \Rightarrow \quad y^2 = 36 \quad \Rightarrow \quad y = \pm 6.$$

So, $|xy| = |x||y| = 9 \cdot 6 = 54$.

3. A regular 2008-gon is located in the Cartesian plane such that $(x_1, y_1) = (p, 0)$ and $(x_{1005}, y_{1005}) = (p + 2, 0)$, where p is prime. The vertices,

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_{2008}, y_{2008}),$$

are arranged in counterclockwise order. Let

$$S = (x_1 + y_1i)(x_3 + y_3i)(x_5 + y_5i) \cdots (x_{2007} + y_{2007}i),$$

$$T = (y_2 + x_2i)(y_4 + x_4i)(y_6 + x_6i) \cdots (y_{2008} + x_{2008}i).$$

Find the minimum possible value of $|S - T|$.

Answer: 2

Solution: The terms in the two products are collectively the 2008th roots of unity, just shifted in the xy -plane.

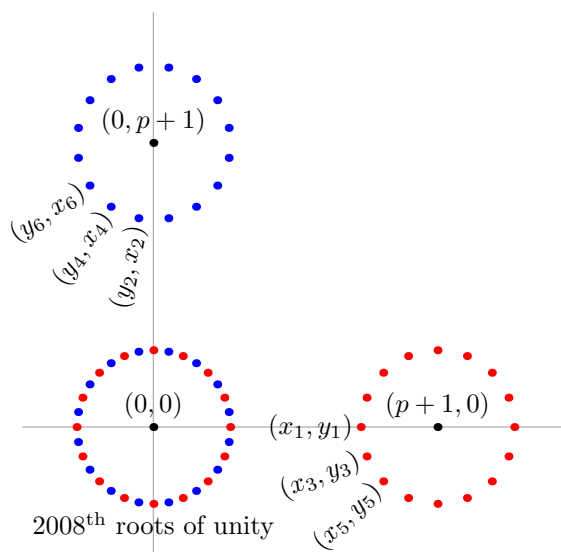


Figure 2: Problem 2-3

The terms in the product S are the 1004th roots of unity, shifted $p + 1$ units to the right. So, where the 1004th roots of unity are the solutions to

$$a^{1004} = 1 \quad \Leftrightarrow \quad a^{1004} - 1 = 0,$$

the complex numbers $x_1 + y_1i, x_2 + y_2i, \dots, x_{2007} + y_{2007}i$ are the solutions to

$$[a - (p + 1)]^{1004} - 1 = 0.$$

Now, S is the product of the roots of this polynomial. By Vieta, that product is

$$S = (p + 1)^{1004} - 1.$$

The terms in the product T are the 2008th roots of unity that are not 1004th roots of unity, shifted $p + 1$ units in the positive real direction, then reflected over the line $x = y$. Another

way of putting this is that the terms in the product T are the 2008th roots of unity that are not 1004th roots of unity, shifted $p+1$ units up the imaginary axis. The 2008th roots of unity that are not 1004th roots of unity are the roots of the equation

$$\frac{b^{2008} - 1}{b^{1004} - 1} = 0 \quad \Leftrightarrow \quad b^{1004} + 1 = 0.$$

So, the terms in the product T are the roots of

$$[b - (p+1)i]^{1004} + 1 = 0.$$

By Vieta, that product is

$$T = [(p+1)i]^{1004} + 1 = (p+1)^{1004}i^{1004} + 1 = (p+1)^{1004} + 1.$$

Now we see that regardless of the value of p ,

$$|S - T| = |[(p+1)^{1004} - 1] - [(p+1)^{1004} + 1]| = |-2| = 2.$$

4. Find the maximum of $x + y$ given that x and y are positive real numbers that satisfy

$$x^3 + y^3 + (x + y)^3 + 36xy = 3456.$$

Answer: 12

Solution: It often helps to substitute symmetric sums for the variables in a problem involving symmetric polynomials. Letting $s = x + y$ and $p = xy$, we see that

$$\begin{aligned} x^3 + y^3 + (x + y)^3 + 36xy &= (x + y)(x^2 - xy + y^2) + s^3 + 36p \\ &= s[(x + y)^2 - 3xy] + s^3 + 36p \\ &= s(s^2 - 3p) + s^3 + 36p \\ &= 2s^3 - 3sp + 36p. \end{aligned}$$

Under our substitution, the given equation becomes

$$2s^3 - 3sp + 36p = 3456.$$

The given equation doesn't really scream for any particular inequality application. Perhaps factoring is the most obvious avenue for exploration. To aide our search, we find the prime factorization of the right-hand side of the equation:

$$2s^3 - 3sp + 36p = 2^7 \cdot 3^3.$$

Looking for useful ratios of coefficients, we eventually spot

$$\left(\frac{36}{-3}\right)^3 = -\frac{2^7 \cdot 3^3}{2}.$$

So, we reorient the equation to take advantage of a difference of cubes factorization:

$$2(s^3 - 2^6 \cdot 3^3) - 3p(s - 12) = 0 \quad \Leftrightarrow \quad 2(s^3 - 12^3) - 3p(s - 12) = 0.$$

Factoring the difference of cubes, we get

$$2(s - 12)(s^2 + 12s + 144) - 3p(s - 12) = (s - 12)(2s^2 + 24s + 288 - 3p) = 0.$$

So, either

- $s = 12$, or
- $2s^2 + 24s + 288 - 3p = 0$.

We need to find the range of solutions to the second equation. In order to do so, we must note a special relationship between s and p . By the Trivial Inequality,

$$(x - y)^2 \geq 0 \quad \Rightarrow \quad (x + y)^2 \geq 4xy,$$

so $s^2 \geq 4p$. There are other ways to derive this relationship as well. We just need to decide to look for constraints on s and p .

Now,

$$2s^2 + 24s + 288 - 3p \geq 2s^2 + 24s + 288 - \frac{3}{4}s^2 = \frac{5}{4}s^2 + 24s + 288.$$

This gives us a one-variable inequality:

$$\frac{5}{4}s^2 + 24s + 288 \leq 0 \quad \Leftrightarrow \quad 5s^2 + 96s + 1152 \leq 0.$$

The discriminant of $5s^2 + 96s + 1152$ is

$$96^2 - 4(5)(1152) = 9216 - 23040 > 0.$$

This means that $5s^2 + 96s + 1152$ is always positively valued, so there are no solutions in real numbers (s, p) to $2s^2 + 24s + 288 - 3p = 0$. Hence, $s = 12$ is the only (and largest) value of $x + y$.

5. While running from an unrealistically rendered zombie, Willy Smithers runs into a vacant lot in the shape of a square, 100 meters on a side. Call the four corners of the lot corners 1, 2, 3, and 4, in clockwise order. For $k = 1, 2, 3, 4$, let d_k be the distance between Willy and corner k . Let
- $d_1 < d_2 < d_4 < d_3$,
 - d_2 is the arithmetic mean of d_1 and d_3 , and
 - d_4 is the geometric mean of d_2 and d_3 .

If d_1^2 can be written in the form $\frac{a - b\sqrt{c}}{d}$, where a, b, c , and d are positive integers, c is square-free, and the greatest common divisor of a, b , and d is 1, find the remainder when $a + b + c + d$ is divided by 2008.

Answer: 1747

Solution: We begin by rewriting conditions (b) and (c) more plainly:

$$\begin{aligned} 2d_2 &= d_1 + d_3, \\ d_4^2 &= d_2d_3. \end{aligned}$$

We note that a d_2d_3 term appears if we rearrange the first of these equations and square:

$$d_1 = 2d_2 - d_3 \quad \Rightarrow \quad d_1^2 = 4d_2^2 - 4d_2d_3 + d_3^2.$$

Combining this with our rewritten condition (3), we have

$$d_1^2 = 4d_2^2 - 4d_4^2 + d_3^2.$$

The squared distances make the algebra of coordinates quite reasonable. We let corner 1 be the origin, so that corner 2 is $(0, 100)$, etc. Willy's position is (x, y) , so from the the last equation

$$x^2 + y^2 = 4[x^2 + (100 - y)^2] - 4[(100 - x)^2 + y^2] + [(100 - x)^2 + (100 - y)^2],$$

which simplifies dramatically to

$$3x - 5y + 100 = 0 \quad \Leftrightarrow \quad y = \frac{3}{5}x + 20.$$

Now, we have a line on which Willy lies. We can substitute this line back most easily into condition (b):

$$2\sqrt{x^2 + (100 - y)^2} = \sqrt{x^2 + y^2} + \sqrt{(100 - x)^2 + (100 - y)^2}$$

which becomes

$$2\sqrt{x^2 + \left(80 - \frac{3}{5}x\right)^2} = \sqrt{x^2 + \left(\frac{3}{5}x + 20\right)^2} + \sqrt{(100 - x)^2 + \left(80 - \frac{3}{5}x\right)^2}.$$

Squaring gets rid of most of the radicals (and requires a bit of bravery), and after combining like terms and simplifying, we get

$$\frac{34}{25}x^2 - 56x + 4400 = \sqrt{\left(\frac{34}{25}x^2 + 24x + 400\right) \left(\frac{34}{25}x^2 - 296x + 16400\right)}.$$

Squaring again and simplifying gives us a cubic equation:

$$17x^3 - 50x^2 - 60000x + 1000000 = 0.$$

We could try plugging in rational roots, but there are so many possibilities that we risk going mad in the process. Fortunately, we can note that were it not for condition (a), conditions (b) and (c) would have the trivial solution where Willy lies in the center of the square. This tells us that $x = 50$ is a root of the cubic. Dividing out $x - 50$, we get

$$17x^2 + 800x - 20000 = 0.$$

Solutions to this quadratic are

$$\frac{-400 \pm 500\sqrt{2}}{17}.$$

We want the positive root, and we can also find $y = \frac{3}{5}x - 20$:

$$(x, y) = \left(\frac{500\sqrt{2} - 400}{17}, \frac{300\sqrt{2} + 100}{17} \right).$$

Thus, d_1^2 is

$$x^2 + y^2 = \frac{50000 - 20000\sqrt{2}}{17}.$$

So, $a + b + c + d = 70019$ and

$$70019 = 34 \cdot 2008 + 1747.$$

The 2008 iTest TOC Round 3 Solutions

1. Find the remainder when $712!$ is divided by 719.

Answer: 718

Solution: Since 719 is prime, we invoke Wilson's Theorem, which doesn't give us what we want, but does give us something that looks a lot like what we want:

$$718! \equiv -1 \pmod{719}.$$

However, we want the modulo 719 residue of $N = 712!$, and so we note that

$$\begin{aligned} 718! &= 718 \cdot 717 \cdot 716 \cdot 715 \cdot 714 \cdot 713 \cdot N \\ &\equiv (-1)(-2)(-3)(-4)(-5)(-6)N \\ &\equiv 720N \equiv N \pmod{719}. \end{aligned}$$

Hence $N \equiv 718! \equiv -1 \pmod{719}$, and the remainder when N is divided by 719 is $719 - 1 = 718$.

2. Note that there are exactly three ways to write the integer 4 as a sum of positive odd integers where the order of the addends matters:

$$\begin{aligned} 1 + 1 + 1 + 1 &= 4, \\ 1 + 3 &= 4, \\ 3 + 1 &= 4. \end{aligned}$$

Let $f(n)$ be the number of ways to write a natural number n as a sum of positive odd integers where the order of the summands matters. Find the remainder when $f(2008)$ is divided by 100.

Answer: 71

Solution: In the process of constructing the possible representations of n (as sums of positive odd integers), we note a one-to-one correspondence between the representations of n that end in 1 and the representations of $n - 1$. Now, looking for a way to relate the representations of n that do not end in 1 to previous constructions, we note that we can subtract 2 from the final addend to get a representation of $n - 2$. The result is that

$$f(n) = f(n - 1) + f(n - 2), \quad n \geq 3.$$

We have the Fibonacci Sequence, and $f(n) = F_n$, where F_n is the n^{th} Fibonacci Number.

Now we must find the remainder when F_{2008} is divided by 100. It might be easiest to find the remainders when F_{2008} is divided by 4 and then by 25.

$$1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, \dots$$

As we see, $F_n \equiv F_{n+6} \pmod{4}$. Since $2008 = 334 \cdot 6 + 4$, $F_{2008} \equiv F_4 \equiv 3 \pmod{4}$.

$$1, 1, 2, 3, 0, 3, 3, 1, 4, 0, \dots$$

Finding the modulo 25 period proves more difficult. Even the modulo 5 period takes a bit of time. As we see above, going out 5 terms triples the modulo 5 residue. So, $F_{n+5} \equiv 3F_n \pmod{5}$, and by Fermat's Little Theorem,

$$F_{n+20} \equiv 3^4 F_n \equiv F_n \pmod{5}.$$

Now, we do a little computation to find that $F_{26} \equiv 18 \pmod{25}$, so $F_{n+25} \equiv 18F_n \pmod{25}$.

$$\begin{aligned} F_{n+50} &\equiv 18^2 F_n \equiv 24 \pmod{25}, \\ F_{n+100} &\equiv 24^2 F_n \equiv 1 \pmod{25}. \end{aligned}$$

So, $F_{2008} \equiv F_8 \equiv 21 \pmod{25}$.

Now we have the system of linear congruence,

$$\begin{aligned} F_{2008} &\equiv 21 \pmod{25}, \\ F_{2008} &\equiv 3 \pmod{4}. \end{aligned}$$

We could solve this system by more algorithmic means, but $21 \equiv 46 \equiv 71 \pmod{25}$, where $71 \equiv 3 \pmod{4}$, so $F_{2008} \equiv 71 \pmod{100}$, and 71 is our answer.

3. Arthur stands on a circle drawn with chalk in a parking lot. It is sunrise and there are birds in the trees nearby. He stands on one of five triangular nodes that are spaced equally around the circle, wondering if and when the aliens will pick him up and carry him to their homeworld. Initially, Arthur faces the clockwise direction (from an overhead view) from the node he is standing on. He flips a fair coin 12 times, each time chanting the name of a nearby star system. Each time he flips a head, he walks around the circle, in the direction he is facing, until he reaches the next node in that direction. Each time he flips a tail, he reverses direction, then walks around the circle until he reaches the next node in that new direction. After 12 flips, Arthur finds himself on the node at which he started. He thinks this is fate, though Arthur is quite mistaken. If a and b are relatively prime positive integers such that a/b is the probability that Arthur flipped exactly 6 heads, find $a + b$.

Answer: 287

Solution: As we do with most combinatorial probability problems, we begin by computing the denominator, because its computation is more plain. Let cw mean clockwise and ccw mean counterclockwise. Getting back to his starting point means Arthur moved cw 1, 6, or 11 times. Since cw and ccw moves are equally probable, the number of outcomes where Arthur gets back to his starting point is

$$\binom{12}{1} + \binom{12}{6} + \binom{12}{11} = 12 + 924 + 12 = 948.$$

As with most discrete probability problems, the numerator is more difficult to compute. We note that the six tails result in the only shifts in direction, so they result in exactly 3 cw and 3 ccw moves. This means that Arthur could not have moved 1 or 11 times cw. So, we must find the number of times he could have made 6 cw and 6 ccw moves while flipping 6 heads. This means he made 3 cw and 3 ccw moves from flipping 6 heads. So, 3 of the heads occurred after an even number of tail flips and 3 after an odd number. For instance,

HHTHHTTTHTH

Here we bold the 3 heads that occur after an even number of tails in the sequence of flips. These flips move Arthur cw. The other 3 move him ccw, so he winds up where he started.

Now, we need to determine

- (a) the number of ways to distribute 3 heads amongst the 4 positions that occur after 0, 2, 4, and 6 tails, and
- (b) the number of ways to distribute 3 heads amongst the 3 positions that occur after 1, 3, and 5 tails.

We use the “stars and bars” model. For (a), there are 4 positions, so we need 3 partitions (bars) dividing up the 3 stars:

$$**||*|$$

is the example of the sequence listed above. There are $\binom{6}{3} = 20$ ways to arrange the stars and bars, representing the ways to arrange the heads according to the tails as in (a). For (b), there are 3 positions, so we need 2 partitions dividing up the 3 stars:

$$**||*$$

is the example of the sequence listed above. There are $\binom{5}{3} = 10$ ways to arrange the stars and bars, representing the ways to arrange the heads according to the tails as in (b).

In total, there are $20 \cdot 10 = 200$ sequences with 6 heads and 6 tails that carry Arthur back to this starting point. So, the probability is

$$\frac{200}{948} = \frac{50}{237},$$

and the answer is $50 + 237 = 287$.

4. Euclid places a morsel of food at the point $(0, 0)$ and an ant at the point $(1, 2)$. Every second, the ant walks one unit in one of the four coordinate directions. However, whenever the ant moves to $(x, \pm 3)$, Euclid’s malicious brother Möbius picks it up and puts it at $(-x, \mp 2)$, and whenever it moves to $(\pm 2, y)$, his cousin Klein puts it at $(\mp 1, y)$. If p and q are relatively prime positive integers such that $\frac{p}{q}$ is the expected number of steps the ant takes before reaching the food, find $p + q$.

Answer: 637

Credit: Problem and solution by Adam Hesterberg, edited by Mathew Crawford and Zachary Abel.

Solution: Let $E_{(a,b)}$ be the expected time for the ant to reach the food from (a, b) . By symmetry, $a = E_{(-1,0)} = E_{(1,0)}$, $b = E_{(0,-1)} = E_{(0,1)}$, $c = E_{(-1,-1)} = E_{(-1,1)} = E_{(1,-1)} = E_{(1,1)}$, $d = E_{(0,-2)} = E_{(0,2)}$, and $e = E_{(-1,-2)} = E_{(-1,2)} = E_{(1,-2)} = E_{(1,2)}$. By considering

the ant's four movement choices from each of these points, we obtain the system of equations

$$\begin{array}{cccccc}
 -3a & & +2c & & & = -4 \\
 & -4b & +2c & +d & & = -4 \\
 a & +b & -3c & & +e & = -4 \\
 & b & & -3d & +2e & = -4 \\
 & & c & +d & -2e & = -4
 \end{array}$$

The solution to this system is

$$a = \frac{380}{29}, b = \frac{512}{29}, c = \frac{432}{29}, d = \frac{588}{29}, e = \frac{608}{29}.$$

The sum of the numerator and denominator of e is $608 + 29 = 637$.

5. It is well-known that the n^{th} triangular number can be given by the formula $n(n+1)/2$. A Pythagorean triple of *square numbers* is an ordered triple (a, b, c) such that $a^2 + b^2 = c^2$. Let a Pythagorean triple of *triangular numbers* (a PTTN) be an ordered triple of positive integers (a, b, c) such that $a \leq b < c$ and

$$\frac{a(a+1)}{2} + \frac{b(b+1)}{2} = \frac{c(c+1)}{2}.$$

For instance, $(3, 5, 6)$ is a PTTN ($6 + 15 = 21$). Here we call both a and b *legs* of the PTTN. Find the smallest natural number n such that n is a leg of *at least* six distinct PTTNs.

Answer: 14

Credit: This problem was inspired by the tinkerings of K. R. S. Sastry, published in the October 1993 *CruX Mathematicorum*.

Solution: Where

$$\frac{n(n+1)}{2} + \frac{b(b+1)}{2} = \frac{c(c+1)}{2},$$

we have

$$n(n+1) = c(c+1) - b(b+1) = c^2 + c - b^2 - b = (c-b)(c+b+1).$$

We note that $c-b$ and $c+b+1$ have opposite parity.

To count the number of ordered pairs (b, c) that we can group with n (to form a PTTN), we consider the prime factorization

$$(c-b)(c+b+1) = 2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \cdots p_k^{e_k} = n(n+1).$$

Here, $c-b$ and $c+b+1$ are both divisors of $n(n+1)$, but their opposite parity restricts the distribution of the powers of 2, which must all be farmed into either the prime factorization of $c-b$ or the prime factorization of $c+b+1$. The number of ordered pairs (b, c) is

$$2(e_2 + 1)(e_3 + 1) \cdots (e_k + 1),$$

but $c - b < c + b + 1$, so the number of ordered pairs (b, c) is $(e_2 + 1)(e_3 + 1) \cdots (e_k + 1)$. We throw out the degenerate case $(b, c) = (0, n)$, and there are $(e_2 + 1)(e_3 + 1) \cdots (e_k + 1) - 1$.

We are looking for $n(n + 1)$ that have at least 6 odd factors, so it's likely that plugging in numbers will not take terribly long. Here, $P(n)$ is the number of PTTNs for which n is a leg:

$n = 2$	\rightarrow	$2 \cdot 3 = 6 = 2^1 \cdot 3^1$	\rightarrow	$P(2) = (1 + 1) - 1$	$= 1,$
$n = 3$	\rightarrow	$3 \cdot 4 = 12 = 2^2 \cdot 3^1$	\rightarrow	$P(3) = (1 + 1) - 1$	$= 1,$
$n = 4$	\rightarrow	$4 \cdot 5 = 20 = 2^2 \cdot 5^1$	\rightarrow	$P(4) = (1 + 1) - 1$	$= 1,$
$n = 5$	\rightarrow	$5 \cdot 6 = 30 = 2^1 \cdot 3^1 \cdot 5^1$	\rightarrow	$P(5) = (1 + 1)(1 + 1) - 1$	$= 3,$
$n = 6$	\rightarrow	$6 \cdot 7 = 42 = 2^1 \cdot 3^1 \cdot 7^1$	\rightarrow	$P(6) = (1 + 1)(1 + 1) - 1$	$= 3,$
$n = 7$	\rightarrow	$7 \cdot 8 = 56 = 2^3 \cdot 7^1$	\rightarrow	$P(7) = (1 + 1) - 1$	$= 1,$
$n = 8$	\rightarrow	$8 \cdot 9 = 72 = 2^3 \cdot 3^2$	\rightarrow	$P(8) = (2 + 1) - 1$	$= 2,$
$n = 9$	\rightarrow	$9 \cdot 10 = 90 = 2^1 \cdot 3^2 \cdot 5^1$	\rightarrow	$P(9) = (2 + 1)(1 + 1) - 1$	$= 5,$
$n = 10$	\rightarrow	$10 \cdot 11 = 110 = 2^1 \cdot 5^1 \cdot 11^1$	\rightarrow	$P(10) = (1 + 1)(1 + 1) - 1$	$= 3,$
$n = 11$	\rightarrow	$11 \cdot 12 = 132 = 2^2 \cdot 3^1 \cdot 11^1$	\rightarrow	$P(11) = (1 + 1)(1 + 1) - 1$	$= 3,$
$n = 12$	\rightarrow	$12 \cdot 13 = 156 = 2^2 \cdot 3^1 \cdot 13^1$	\rightarrow	$P(12) = (1 + 1)(1 + 1) - 1$	$= 3,$
$n = 13$	\rightarrow	$13 \cdot 14 = 182 = 2^1 \cdot 7^1 \cdot 13^1$	\rightarrow	$P(13) = (1 + 1)(1 + 1) - 1$	$= 3,$
$n = 14$	\rightarrow	$14 \cdot 15 = 210 = 2^1 \cdot 3^1 \cdot 5^1 \cdot 7^1$	\rightarrow	$P(14) = (1 + 1)(1 + 1)(1 + 1) - 1 = 7.$	

So, $n = 14$ is the smallest natural number that is the leg of at least 6 (a total of 7) PTTNs.

The 2008 iTest TOC Round 4 Solutions

1. Yatta and Yogi play a game in which they begin with a pile of n stones. The players take turns removing 1, 2, 3, 5, 6, 7, or 8 stones from the pile. That is, when it is a player's turn to remove stones, that player may remove from 1 to 8 stones, but *cannot* remove exactly 4 stones. The player who removes the last stone *loses*. Yogi goes first and finds that he has a winning position, meaning that so long as he plays perfectly, Yatta cannot defeat him. For how many positive integers n from 100 to 2008 inclusive is this the case?

Answer: 1615

Solution: We identify both winning and losing positions by starting at “the end” (small piles) and working backward through the possible values of n . Consider the following table of winning/losing positions:

<u>Winning Positions</u>	<u>Losing Positions</u>
	1
2	
3	
4	
	5
6	
7	
8	
9	
10	
11	
12	
13	
	14

From each of the winning positions, a player can force his opponent into a losing position. We bold 9 because 9 cannot lead to 5, but 9 *can* lead to 1. This is a key moment in the pattern of winning and losing moves, allowing a full stretch of 8 winning positions before the entire “cycle” begins again after 13. In general, we find that when m stones remain, the position is a *losing* position if and only if

$$m \equiv 1 \pmod{13},$$

$$m \equiv 5 \pmod{13}.$$

The rest of the positions are all winning positions. In the chart above, ignore 14, and view 13 as 0 (mod 13). Now, make sure you see that from each losing position, a player has no choice but to leave his opponent with a winning position, and that from each winning position, a player can leave his opponent with a losing position.

Now, we count. It is easier to count the number of losing positions, then subtract from $2008 - 100 + 1 = 1909$ (which is the total values of n from 100 to 2008 inclusive). First, we

note that

$$\begin{aligned}100 &= 7 \cdot 13 + 9, \\2008 &= 154 \cdot 13 + 6.\end{aligned}$$

So, we count the integers congruent to 1 (mod 13) from $8 \cdot 13 + 1$ up to $154 \cdot 13 + 1$, and then the integers congruent to 5 (mod 13) from $8 \cdot 13 + 5$ up to $154 \cdot 13 + 5$, and we get

$$(154 - 8 + 1) + (154 - 8 + 1) = 2(147) = 294.$$

Thus, there are $1909 - 294 = 1615$ total winning positions as n ranges from 100 through 2008 inclusive.

2. Let A the number of 12-digit words that can be formed from the alphabet $\{0, 1, 2, 3, 4, 5, 6\}$ if each pair of neighboring digits must differ by exactly 1. Find the remainder when A is divided by 2008.

Credit: A more general problem with a smaller alphabet appeared on the 1987 IMO shortlist. Solution by Zach Abel.

Answer: 1392

Solution: Let $a_0(n)$ denote the number of legal words of length n that end with 0, and likewise for $a_1(n)$ through $a_6(n)$. Note that $a_i(n) = a_{6-i}(n)$, so we really only need to calculate the values $a_0(n)$ through $a_3(n)$. We easily derive the following recursions:

$$\begin{aligned}a_0(n+1) &= a_1(n) \\a_1(n+1) &= a_0(n) + a_2(n) \\a_2(n+1) &= a_1(n) + a_3(n) \\a_3(n+1) &= 2a_2(n),\end{aligned}$$

where the last one follows from $a_3(n+1) = a_2(n) + a_4(n) = 2a_2(n)$.

So, we start tabulating some values; note that $a_i(1) = 1$, as the word may start with any character in $\{0, 1, 2, 3, 4, 5, 6\}$:

	1	2	3	4	5	6	...	n
0	1	1	2	3	6	10	...	$a_0(n)$
1	1	2	3	6	10	20	...	$a_1(n)$
2	1	2	4	7	14	24	...	$a_2(n)$
3	1	2	4	8	14	28	...	$a_3(n)$

We could straightforwardly continue tabulating to column $n = 12$. However, we could save some work by noticing a pattern in the table. The odd columns seem to have the form

$$\{a_0(2t-1), a_1(2t-1), a_2(2t-1), a_3(2t-1)\} = \{x_t - y_t, x_t, x_t + y_t, x_t + y_t\}, \quad (1)$$

while the even columns look like

$$\{a_0(2t), a_1(2t), a_2(2t), a_3(2t)\} = \{x_t, 2x_t, 2x_t + y_t, 2x_t + 2y_t\}. \quad (2)$$

This pattern indeed continues. If we assume the $(2t - 1)$ st column has the form of (1), then by the recurrences, we find

$$\begin{aligned} a_0(2t) &= a_1(2t - 1) = x_t, \\ a_1(2t) &= a_0(2t - 1) + a_2(2t - 1) = 2x_t, \\ a_2(2t) &= a_1(2t - 1) + a_3(2t - 1) = 2x_t + y_t, \\ a_3(2t) &= a_0(2t - 1) + a_2(2t - 1) = 2x_t + 2y_t, \end{aligned}$$

which agree with (2), and furthermore,

$$\begin{aligned} a_0(2t + 1) &= a_1(2t) = 2x_t = (3x_t + y_t) - (x_t + y_t), \\ a_1(2t + 1) &= a_0(2t) + a_2(2t) = 3x_t + y_t = (3x_t + y_t), \\ a_2(2t + 1) &= a_1(2t) + a_3(2t) = 4x_t + 2y_t = (3x_t + y_t) + (x_t + y_t), \\ a_3(2t + 1) &= a_0(2t) + a_2(2t) = 4x_t + 2y_t = (3x_t + y_t) + (x_t + y_t), \end{aligned}$$

which agree with (1) with $x_{t+1} = 3x_t + y_t$ and $y_{t+1} = x_t + y_t$. Thus, in order to compute the 12th column of the above table, we must only calculate x_6 and y_6 :

$$\begin{array}{lll} (x_1, y_1) = (1, 0) & (x_2, y_2) = (3, 1) & (x_3, y_3) = (10, 4) \\ (x_4, y_4) = (34, 14) & (x_5, y_5) = (116, 48) & (x_6, y_6) = (396, 164). \end{array}$$

The total number of words with length 12 is

$$\begin{aligned} &a_0(12) + a_1(12) + \cdots + a_6(12) \\ &= (x_6) + (2x_6) + (2x_6 + y_6) + (2x_6 + 2y_6) + (2x_6 + y_6) + (2x_6) + (x_6) \\ &= 12x_6 + 4y_6 \\ &= 12(396) + 4(164) \\ &= 5408. \end{aligned}$$

Finally, $5408 = 2 \cdot 2008 + 1392$.

3. For how many integers $1 \leq n \leq 9999$ is there a solution to the congruence

$$\phi(n) \equiv 2 \pmod{12},$$

where $\phi(n)$ is the Euler phi-function?

Credit: Solution by Adam Hesterberg

Answer: 13

Solution: If $\prod p_i^{a_i}$ is the prime factorization of n , then

$$\phi(n) = \prod ((p_i^{a_i-1})(p_i - 1)).$$

We consider possible prime factors of $n \pmod{12}$:

- If n has no prime factors, then $n = 1$ and $\phi(1) = 1 \not\equiv 2 \pmod{12}$. Hence n has a prime factor.

- There are no primes congruent to 0, 4, 6, 8, 9, or 10 (mod 12).
- If n has a prime factor p_i congruent to 1 or 5 (mod 12), then $p_i - 1 \equiv 0 \pmod{4}$, so $\phi(n) \not\equiv 2 \pmod{12}$
- If n has a prime factor p_i congruent to 7 (mod 12), then $p_i - 1 \equiv 0 \pmod{3}$, so $\phi(n) \not\equiv 2 \pmod{12}$
- If n has more than one odd prime factor, then for each one $p_i - 1 \equiv 0 \pmod{2}$, so $\phi(n) \not\equiv 0 \pmod{12}$. Hence n has at most one odd prime factor.
- n does not have a factor of 3^2 , since then $n \equiv 0 \pmod{3}$. If $n = 3$, $\phi(n) = 2$. Otherwise, if $3|n$, then $n = 2^k$ for some $k > 0$, so $\phi(n) = 2^k$. Since $\phi(n) \equiv 2 \pmod{12}$, $k = 1$. Hence if $3|n$, then $n = 3$ or $n = 6$.
- If n has no odd prime factors, then $n = 2^k$ and $\phi(n) = 2^{k-1} \equiv 2 \pmod{12}$, so $k = 2$.
- This last case is the most fruitful. If n has a prime factor p_i congruent to 11 (mod 12), then $n = 2^k p_i^j$ for some $k \geq 0$, $j \geq 1$. Then

$$\phi(n) = \phi(2^k)(p_i - 1)p_i^{j-1}.$$

But $p_i - 1 \equiv -2 \pmod{12}$ and $p_i \equiv -1 \pmod{12}$, so

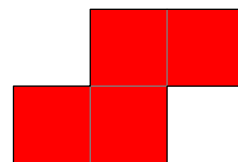
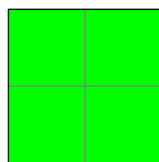
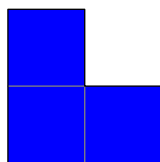
$$\phi(n) \equiv 2\phi(2^k)(-1)^j \pmod{12}.$$

This is congruent to 2 iff j is even and $\phi(2^k) \equiv 0$, that is, $k = 0$ or $k = 1$. Also, for any prime $p_i \equiv 11 \pmod{12}$, $p_i^4 \geq 10000$, so $j = 2$. An exhaustive search of such values reveals exactly the following:

$$\begin{aligned} 121 &= 11^2, \\ 242 &= 2 \cdot 11^2, \\ 529 &= 23^2, \\ 1058 &= 2 \cdot 23^2, \\ 2209 &= 47^2, \\ 4418 &= 2 \cdot 47^2, \\ 3481 &= 59^2, \\ 6962 &= 2 \cdot 59^2, \\ 5041 &= 71^2, \\ 6889 &= 83^2. \end{aligned}$$

From the last three cases, we find $2 + 1 + 10 = 13$ values of n .

4. Each of the 24 students in Mr. Friedman's class cut up a 7×7 grid of squares while he read them short stories by Mark Twain. While not all of the students cut their squares up in the same way, each of them cut their 7×7 square up into at most the three following types (shapes) of pieces



Let a , b , and c be the number of total pieces of each type from left to right respectively after all $24\ 7 \times 7$ squares are cut up. How many ordered triples (a, b, c) are possible?

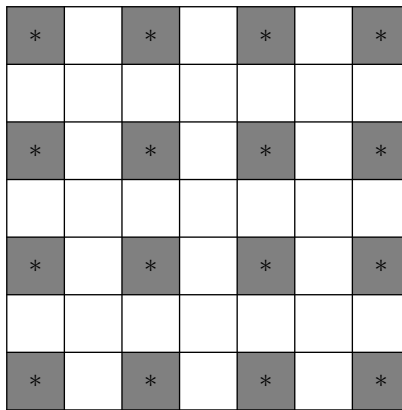
Answer: 25

Solution: First, we have two ways to add up the number of 1×1 squares in the 7×7 squares:

$$3a + 4b + 4c = 24 \cdot 7^2. \tag{3}$$

Now, we go hunting for more limiting information.

Examining each individual square, we note that each of the three types of pieces can contain only one of the squares marked with an asterisk (*):



There are 16 asterisks per square, so there must be *at least* 16 pieces of the three types per square:

$$a + b + c \geq 16 \cdot 24 = 384. \tag{4}$$

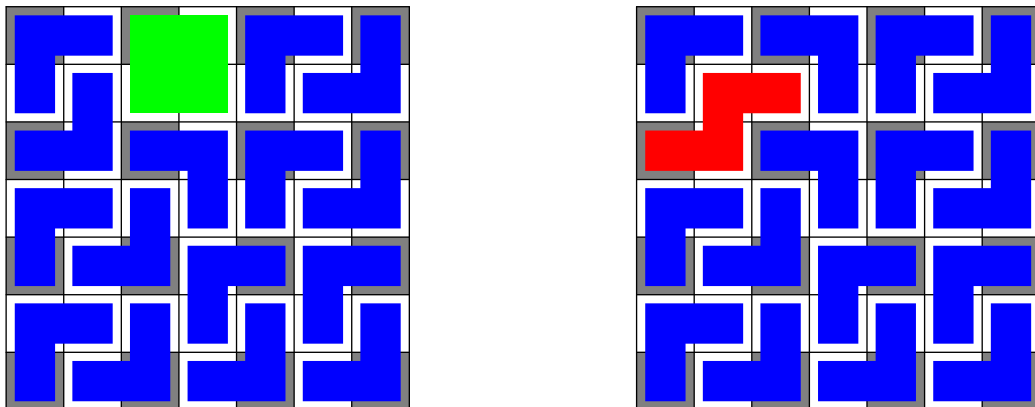
Now, subtracting 3 times Inequality 4 from Equation 3, we get

$$b + c \leq 24.$$

But clearly $b + c \geq 24$, because there must be one piece of either the second or third type in *each* of the $24\ 7 \times 7$ squares purely based on square counts:

$$15 \cdot 3 + 1 \cdot 4 = 7^2.$$

Now, we need to see if a 7×7 square can be made up if 15 of the first piece type, and 1 each of the second or third piece types:



Expressing the piece counts of the students individually, they look like either $(15, 1, 0)$ or $(15, 0, 1)$. Adding these together for all 24 students, we get possible ordered triples $(360, n, 24 - n)$, where n ranges from 0 to 24, for a total of 25 possible ordered triples (a, b, c) .

5. For positive integers $m, n \geq 3$, let $h(m, n)$ be the maximum (finite) number of intersection points between a simple m -gon and a simple n -gon. (Note: a polygon is *simple* if it does not intersect itself.) Evaluate

$$\sum_{m=3}^6 \sum_{n=3}^{12} h(m, n).$$

Answer: 1140

Credit: Problem and solution by Zachary Abel.

Solution: We'll work with the polygons $A = A_1A_2 \dots A_m$ and $B = B_1B_2 \dots B_n$. By slightly perturbing the vertices of A and B if necessary (without losing any intersection points, and perhaps even gaining some!), we may assume that the $m+n$ vertices are in *general position* in the plane, i.e. that no 3 of them are collinear. There are five cases to consider, based (mostly) on the parity of m and n .

- Case 1: m and n both even. Each edge of A intersects each edge of B at most once, so there are at most mn intersection points total. This bound is indeed achievable, using two “clothespin” polygons P_m and P_n as illustrated in Figure 3.
- Case 2: m even, n odd. The following lemma is useful:

Lemma 1. *For a simple n -gon B , any line ℓ not passing through any vertex B_i intersects B an even number of times.*

Proof. Line ℓ divides the plane into two half-planes, which we'll refer to as regions I and II ; suppose B_1 is in region I . Line ℓ intersects segment B_iB_{i+1} if and only if B_i and B_{i+1} are on different sides of ℓ , so it inductively follows that the polygonal path $B_1B_2 \dots B_i$ intersects ℓ an even number of times if and only if B_i is in region I . As path $B_1B_2 \dots B_nB_1 = B_1B_2 \dots B_nB_{n+1}$ certainly has its endpoint B_{n+1} in region I , it intersects ℓ an even number of times, QED. \square

It follows from this Lemma that if n is odd, each edge of A intersects at most $n - 1$ edges of B , i.e. that there are at most $m(n - 1)$ total intersection points. And indeed, as illustrated in Figure 4, this number may be achieved using a “clothespin” polygon P_m and an “alternating triangle” T_n as shown.

- Case 3: m odd, n even. This is equivalent to Case 2; we have $h(m, n) = (m - 1)n$.
- Case 4: m and n both odd, $\min\{m, n\} = 3$. As $h(m, n) = h(n, m)$, we may assume that $3 = m \leq n$. By Lemma 1, each edge of B can intersect at most 2 edges of A , so we have an upper bound of $2n$. This bound is reached by intersecting a triangle T_3 and an alternating triangle T_n as shown in Figure 5.
- Case 5: m and n both odd, $\min\{m, n\} = 5$. As in Case 4, suppose $5 = m \leq n$. Each edge of B intersects at most 4 edges of A , so we have an upper bound of $4n$. However, I'll show that the maximum achievable bound is actually $h(5, n) = 4n - 2$. The argument will make use of the following Lemmata:

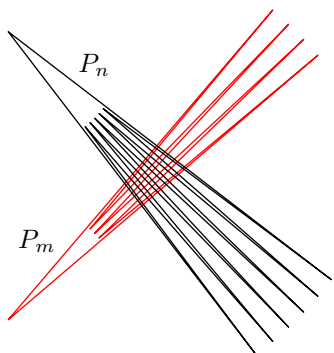


Figure 3: Two clothespin polygons intersecting at mn points.

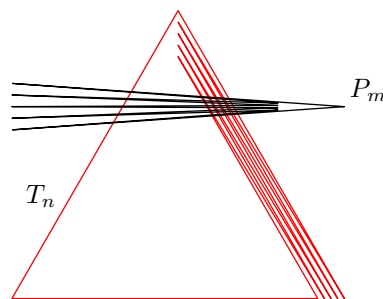


Figure 4: Clothespin and alternating triangle with $m(n - 1)$ intersections.

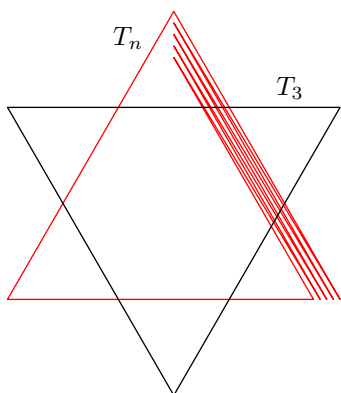


Figure 5: Triangle and alternating triangle; $2n$ intersections.

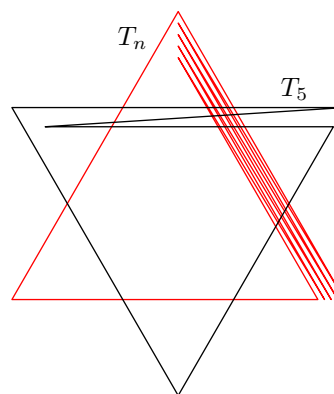


Figure 6: Two alternating triangles, $4n - 2$ intersection points.

Lemma 2. For all integers m and n , $h(m, n)$ is even.

Proof. This is similar to the proof of Lemma 1. Polygon A divides the plane into its interior (region I) and its exterior (region II) by the Jordan curve theorem. Think of B as a directed, cyclic path, and note that at each intersection, the path switches from region I to region II or vice-versa; this is guaranteed by the assumption that no three vertices in A or B are collinear. As path B must end in the same region from which it began, there must be an even number of switches, i.e. an even number of intersection points between A and B . \square

Lemma 3. Say that pentagon $A_1 \cdots A_5$ has an i -cut line if there exists a line that has A_{i-1} and A_{i+1} on one side and the other three vertices on the other side. Equivalently, this i -cut line intersects all edges of A except $A_{i-2}A_{i+2}$. Then there does not exist a simple pentagon with an i -cut line for each $1 \leq i \leq 5$.

Proof. Suppose the vertices are V_1, \dots, V_5 , though not necessarily in that order. There are three types of vertex configurations to consider.

If the convex hull of the vertices has all five points, then A must simply be a convex pentagon. So any line intersects A in at most 2 places, i.e. A has no i -cut lines.

If the convex hull of the vertices has four points, then we have a convex quadrilateral $V_1V_2V_3V_4$ with an interior point V_5 ; without loss of generality, V_5 is in the triangle bounded by V_1V_3 , V_2V_3 , and V_2V_4 as in Figure 7. In this Figure, edge V_iV_j is drawn dashed if there exists a line separating V_i and V_j from the other three vertices; equivalently, if segment V_iV_j does not intersect the interior of the triangle formed by the other three vertices. In order for A to have five cut lines, the five diagonals of A must all be dashed, i.e. the five edges of A must include all of the solid edges. However, as the solid edges intersect each other, this would make A a non-simple polygon, contradiction.

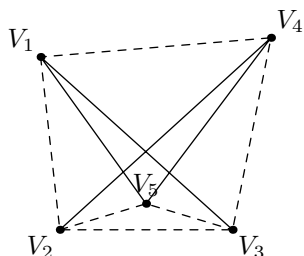


Figure 7: Convex hull of A 's vertices contains four points. Dashed segments have separating lines.

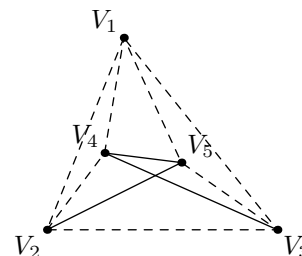


Figure 8: Convex hull of A 's vertices contains three points. Dashed segments have separating lines.

Finally, if the convex hull of the vertices has three points, the configuration without loss of generality is as illustrated in Figure 8. Again, the edges V_iV_j with a separating line are drawn dashed. As before, the solid edges cross, so A does not exist in this configuration. This exhausts the cases, so we're done. \square

To prove that $h(5, n) \leq 4n - 2$ for n odd, it suffices by Lemma 2 to show that $h(5, n) < 4n$. If pentagon A and n -gon B intersect at $4n$ points, then each edge of B intersects exactly 4 of the 5 edges of A . Lemma 3 above guarantees that there is an index i so that A has no i -cut line. This means that any line that intersects four of A 's edges must intersect segment $A_{i+2}A_{i+3}$, which implies that all n edges of B intersect this segment. But as n is odd, segment $A_{i+2}A_{i+3}$ can intersect B at most $n - 1$ times by Lemma 1. This is the desired contradiction.

Finally, by intersecting two alternating triangles T_5 and T_n as illustrated in Figure 6, we may conclude that $h(5, n) = 4n - 2$.

Having concluded the casework, we may now simply add up the answer:

$$\begin{aligned}\sum_{m=3}^6 \sum_{n=3}^{12} h(m, n) &= \sum_{i=2}^3 \sum_{j=2}^6 (h(2i, 2j) + h(2i - 1, 2j) + h(2i, 2j - 1)) \\ &\quad + h(3, 3) + h(3, 5) + h(5, 3) + h(5, 5) + \sum_{j=4}^6 (h(3, 2j - 1) + h(5, 2j - 1)) \\ &= \sum_{i=2}^3 \sum_{j=2}^6 (12ij - 4i - 4j) + 6 + 10 + 10 + 18 + \sum_{j=4}^6 (12j - 8) \\ &= 940 + 44 + 156 \\ &= 1140.\end{aligned}$$

The 2008 iTest TOC Round 5 Solutions

1. Let a , b , c , and d be positive real numbers such that $abcd = 17$. Let m be the minimum possible value of

$$a^2 + b^2 + c^2 + a(b + c + d) + b(c + d) + cd.$$

Compute $\lfloor 17m \rfloor$.

Credit: Solution by Adam Hesterberg, edited by Mathew Crawford.

Answer: 594

Solution: This problem certainly looks like one where we might try to apply the Arithmetic Mean-Geometric Mean Inequality (AM-GM), but a straight-forward application doesn't give us what we want because because the nine-term expression, when expanded, has fewer d terms (the lack of a d^2 term). We must "rebalance" the expression ourselves by manipulating constants in the expression. We rewrite the given expression as

$$a^2 + b^2 + c^2 + ab + ac + bc + a\left(\frac{d}{2}\right) + a\left(\frac{d}{2}\right) + b\left(\frac{d}{2}\right) + b\left(\frac{d}{2}\right) + c\left(\frac{d}{2}\right) + c\left(\frac{d}{2}\right).$$

Now, by AM-GM on the equivalent twelve-term expression, we get

$$a^2 + b^2 + c^2 + ab + ac + bc + (2a + 2b + 2c)\frac{d}{2} \geq 12 \sqrt[12]{\frac{a^6 b^6 c^6 d^6}{2^6}}.$$

Hence the given expression is at least $12\sqrt{\frac{17}{2}}$. This value is attainable when

$$a = b = c = \frac{d}{2} = \sqrt[4]{\frac{17}{2}},$$

so it is indeed the minimum m . Then

$$\lfloor 17m \rfloor = \left\lfloor 204\sqrt{\frac{17}{2}} \right\rfloor.$$

The integer part of this expression can be calculated straightforwardly by rewriting it as $\lfloor \sqrt{204^2 \frac{17}{2}} \rfloor$ and noting that $594^2 < 204^2 \frac{17}{2} = 353736 < 595^2$. Hence $\lfloor 17m \rfloor = 594$.

2. Let N be the smallest natural number that, when written to its left, forms an integer with twice as many digits that is a perfect square. Find the remainder when N is divided by 1000.

Answer: 496

Solution: Let $10^k < N < 10^{k+1}$, so that N is a $k + 1$ digit number. Then, when we place N to the right of N , the resulting integer is

$$(10^{k+1} + 1)N.$$

We use the actual (minimum) value of N here as an example:

$$(10^{11} + 1)13223140496 = 1322314049613223140496 = 36363636364^2.$$

The far left-hand side of this problem is the product of two unequal integers. So, the prime factorizations of these unequal integers are not the same. This means that the larger one ($10^{k+1} + 1$) must include some prime to the second power in its factorization.

Now we hunt for a small k such that $10^{k+1} \equiv 0 \pmod{p^2}$ for some prime p . The fact that 11 is a divisor of each 11, 1001, 100001, etc., points us toward looking for an integer in the sequence

$$\frac{11}{11} = 1,$$

$$\frac{1001}{11} = 91,$$

$$\frac{100001}{11} = 9091,$$

$$\frac{10000001}{11} = 909091,$$

⋮

that is a multiple of 11. Due to the fact that the differences between the consecutive terms are all of the form

$$9 \cdot 10^{2j+1} \equiv 9 \cdot (-1)^{2j+1} \equiv -9 \equiv 2 \pmod{11} \quad (j = 0, 1, 2, 3, \dots),$$

we need not look all that far:

$$\frac{9090909091}{11} = 826446281.$$

However, this is a 9-digit integer, and N must be a $k = 11$ -digit integer since

$$11 \cdot 9090909091 = 100000000001$$

is a $k + 1 = 12$ -digit integer. So, we multiply 826446281 by as small a perfect square as possible to get

$$4^2 \cdot 826446281 = 13223140496.$$

The remainder when 13223140496 is divided by 1000 is 496.

For good measure, we show the important $10^k + 1$ prime factorizations, without getting into all the different algebraic and number sense methods for hunting these down (they are not

particularly complex, but require several pages to write about):

$$\begin{aligned}
 11 &= 11^1, \\
 101 &= 101^1, \\
 1001 &= 7^1 \cdot 11^1 \cdot 13^1, \\
 10001 &= 73^1 \cdot 137^1, \\
 100001 &= 11^1 \cdot 9091^1, \\
 1000001 &= 101^1 \cdot 9901^1, \\
 10000001 &= 11^1 \cdot 909091^1, \\
 100000001 &= 17^1 \cdot 5882353, \\
 1000000001 &= 7^1 \cdot 11^1 \cdot 13^1 \cdot 19^1 \cdot 52579^1, \\
 10000000001 &= 101^1 \cdot 3541^1 \cdot 27961^1.
 \end{aligned}$$

3. The 360 volumes of the *Encyclopedia Galactica* are out of order in the library. Fortunately for the librarian, the books are numbered. Due to his religion, which holds both encyclopedias and the concept of parity in high esteem, the librarian can only rearrange the books two at a time, and then only by switching the position of an even numbered volume with that of an odd numbered volume. Find the minimum number of such transpositions sufficient to get the books back into ordinary sequential order, regardless of the starting positions of the books. (Find the minimum number of transpositions in the worst-case scenario.)

Credit: Based on a problem from the Leningrad Mathematical Olympiad.

Answer: 449

Solution: Before we get into a proof, we do a little modeling of book arrangements. Thinking of each volume as a vertex gives us a little insight. Consider a set of distinct volumes $\{V_1, V_2, V_3, \dots, V_k\}$ such that V_1 is where V_2 should be, V_2 is where V_3 should be, V_3 is where V_4 should be, etc. We call this set a *cycle*. The initial arrangement of the 360 volumes is a set of cycles, and the librarian is trying to rearrange them so that there are 360 cycles of 1 volume each.

Now, we imagine what happens when the librarian switches two volumes. If you need to, draw examples of the following to see why each occurs:

- When the volumes belong to the same cycle, then the cycle breaks into two cycles.
- When the volumes belong to different cycles, switching them fuses the two cycles into a single cycle.

Next, we explore the worst-case scenario according to a series of breaks and fusions. Let an n -cycle be a cycle with exactly n total distinct volumes. Let a *purely even* cycle be a cycle with only even-numbered volumes, and similarly let a *purely odd* cycle be a cycle with only odd-numbered volumes. Let x be the number of purely even cycles of length greater than 1, and y be the number of purely odd cycles of length greater than 1. Then, $x \leq 90$ and $y \leq 90$. Without loss of generality, let $x \leq y$ (and we construct our final example this way).

We must fuse each of the y odd cycles with other cycles before those odd volumes can be broken down into 1-cycles (as they would be in their final positions). This requires at most 90 fusions. Note that since $x \leq y$, this can be done in such a way as to eliminate (through fusion) all the purely even cycles.

With no purely even or odd cycles left, we can put the volumes in order by breaking 1-cycles off larger mixed cycles. This takes at most 359 transpositions of volumes. All in all, the whole process can be complete in no more than $90 + 359 = 449$ transpositions.

Finally, we find an example of a sequence of volumes that requires 449 switches. We need 359 switches to separate books into 1-cycles, and 90 switches to combine the purely even and purely odd cycles. Those 90 switches could come from combining 90 purely odd 2-cycles with a single purely even 180-cycle. It would then require 359 switches to book the books into order (360 1-cycles).

4. Let

$$f(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{1}{n-k} \binom{n-k}{k},$$

for each positive integer n . If $|f(2007) + f(2008)| = a/b$ for relatively prime positive integers a and b , find the remainder when a is divided by 418.

Credit: Written by Mathew Crawford, this problem is based on a 1992 IMO shortlist problem.
Solution: by Adam Hesterberg.

Answer: 23

Solution: We explicitly calculate $b = 2007 \cdot 2008$ and $a = 6023 = 14 \cdot 418 + 171$.

Let $g(n) = nf(n)$. We first simplify $g(n)$ somewhat:

$$\begin{aligned}
 g(n) &= n \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{1}{n-k} \binom{n-k}{k} \\
 &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k} \binom{n-k}{k} \\
 &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{k}{n-k} \binom{n-k}{k} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \\
 &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{k}{n-k} \frac{n-k}{k} \binom{n-k-1}{k-1} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \\
 &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k-1}{k-1} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k}
 \end{aligned}$$

We claim that $g(n) = g(n-1) - g(n-2)$. Indeed, if we shift some indices and apply Pascal's

identity $\left(\binom{a}{b} + \binom{a}{b+1}\right) = \binom{a+1}{b+1}$ to $g(n)$, we get:

$$\begin{aligned}
 g(n) &= \left(\sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k-1}{k-1} \right) + \left(\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \right) \\
 &= \left(\sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k-2}{k-1} + \sum_{k=2}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k-2}{k-2} \right) \\
 &\quad + \left(\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k-1}{k} + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k-1}{k-1} \right) \\
 &= \left(\sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k-2}{k-1} + \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k-1}{k} \right) \\
 &\quad + \left(\sum_{k=1}^{\lfloor (n-2)/2 \rfloor} -(-1)^k \binom{n-k-3}{k-1} + \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} -(-1)^k \binom{n-k-2}{k} \right) \\
 &= \left(\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-k-2}{k-1} + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-k-1}{k} \right) \\
 &\quad - \left(\sum_{k=1}^{\lfloor (n-2)/2 \rfloor} (-1)^k \binom{n-k-3}{k-1} + \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (-1)^k \binom{n-k-2}{k} \right) \\
 &= g(n-1) - g(n-2),
 \end{aligned}$$

as claimed. Then $g(1) = 1$ and $g(2) = -1$ determine $g(n)$, which repeats mod 6: $g(3+6k) = -2$, $g(4+6k) = -1$, $g(5+6k) = 1$, $g(6k) = 2$, $g(1+6k) = 1$, and $g(2+6k) = -1$. Hence

$$f(2007) = \frac{g(2007)}{2007} = -\frac{2}{2007}$$

and $g(2008) = -\frac{1}{2008}$, so

$$|f(2007) + f(2008)| = \frac{6023}{2007 \cdot 2008},$$

as claimed. The remainder when 6023 is divided by 1000 is 23.

5. Two squares of side length 2 are glued together along their boundary so that the four vertices of the first square are glued to the midpoints of the four sides of the other square, and vice versa. This gluing results in a convex polyhedron. If the square of the volume of this polyhedron is written in simplest form as $\frac{a+b\sqrt{c}}{d}$, what is the value of $a + b + c + d$?

Answer: 27

Credit: Problem and solution by Zachary Abel.

Solution: Part of the difficulty of this problem is visualizing the polyhedron; it is illustrated in Figure 9. (Note: by *Alexandrov's Theorem*, this polyhedron is unique.) The common boundary of the two original squares is shown in bold. Label the vertices B_i and T_i and the

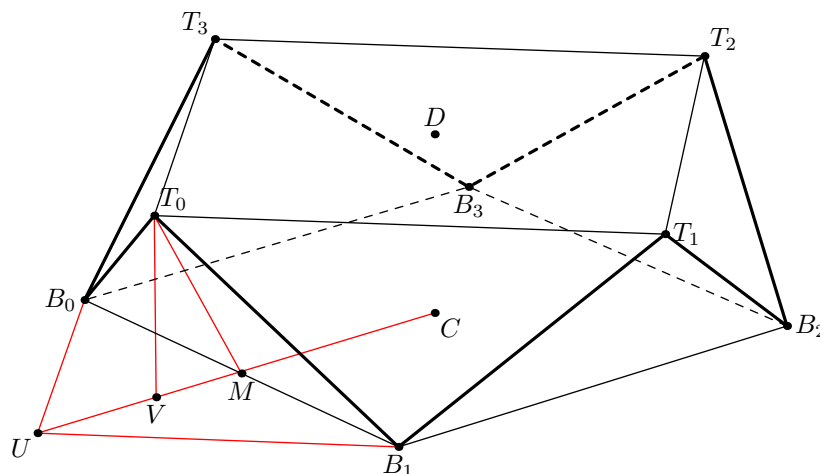


Figure 9: Problem 5-5.

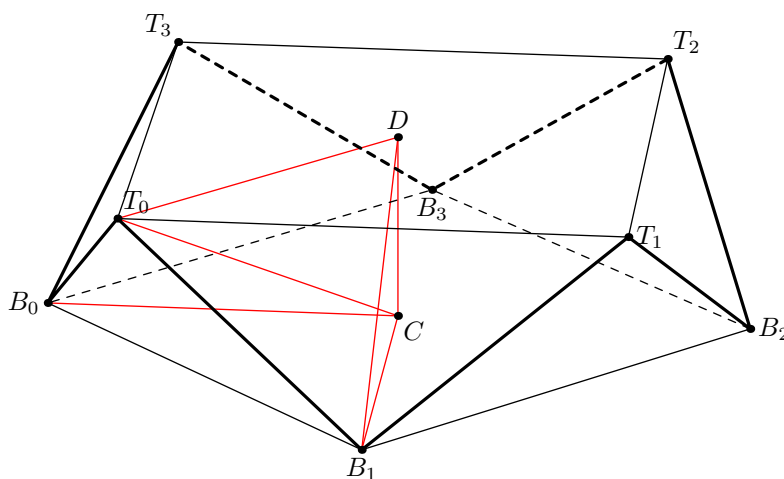


Figure 10: Decomposing the solid into tetrahedra.

centers of the top and bottom squares C and D as illustrated. Project T_0 onto the base plane at V and let polygon $B_0T_0B_1$ unfold to the base plane at B_0UB_1 . Finally, M is the midpoint of B_0B_1 .

We have $CV = DT_0 = 1$, so $MV = 1 - \frac{\sqrt{2}}{2}$. Also, $T_0M = UM = \frac{\sqrt{2}}{2}$, so

$$h = T_0V = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 - \left(1 - \frac{\sqrt{2}}{2}\right)^2} = \sqrt{\sqrt{2} - 1}$$

Now, note that the solid may be divided into 16 tetrahedra as follows: 8 tetrahedra congruent to $B_0B_1CT_0$ and 8 tetrahedra congruent to T_0B_1CD . (See Figure 10.)

The area of B_0B_1C is $\frac{1}{2}$, so the volume of $B_0B_1CT_0$ is $\frac{h}{6}$. Next, the area of T_0CD is $\frac{1}{2}$ and the distance from B_1 to plane T_0CD is $\frac{\sqrt{2}}{2}$, so the volume of T_0B_1CD is $\frac{h\sqrt{2}}{12}$.

The total volume of the solid is thus

$$8h \cdot \left(\frac{1}{6} + \frac{\sqrt{2}}{12} \right) = \frac{2}{3} (2 + \sqrt{2}) \sqrt{\sqrt{2} - 1} = \frac{2\sqrt{2}}{3} \sqrt{\sqrt{2} + 1} = \sqrt{\frac{8 + 8\sqrt{2}}{9}}.$$

The answer is $8 + 8 + 2 + 9 = 27$.

The 2008 iTest TOC Round 6 Solutions

1. Let

$$X = \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} + \cdots + \cos \frac{2006\pi}{7} + \cos \frac{2008\pi}{7}.$$

Compute $|\lfloor 2008X \rfloor|$.

Answer: 1004

Solution: The 7th roots of unity are $\cos \frac{2k\pi}{7} + i \sin \frac{2k\pi}{7}$, where $0 \leq k < 7$. By Vieta, these roots of $x^7 - 1$ add up to 0. Thus, we know that both

$$\cos 0 + \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} + \cos \frac{8\pi}{7} + \cos \frac{10\pi}{7} + \cos \frac{12\pi}{7} = 0,$$

$$\sin 0 + \sin \frac{2\pi}{7} + \sin \frac{4\pi}{7} + \sin \frac{6\pi}{7} + \sin \frac{8\pi}{7} + \sin \frac{10\pi}{7} + \sin \frac{12\pi}{7} = 0.$$

More generally, for any integer n ,

$$\sum_{k=0}^6 \cos \frac{(14n + 2k)\pi}{7} = 0.$$

Now we note that

$$\begin{aligned} X &= \sum_{n=0}^{142} \left(\sum_{k=0}^6 \cos \frac{(14n + 2k)\pi}{7} \right) - \cos 0 + \cos \frac{2002\pi}{7} + \cos \frac{2004\pi}{7} + \cos \frac{2006\pi}{7} + \cos \frac{2008\pi}{7} \\ &= \sum_{n=0}^{142} 0 - \cos 0 + \cos \frac{2002\pi}{7} + \cos \frac{2004\pi}{7} + \cos \frac{2006\pi}{7} + \cos \frac{2008\pi}{7} \\ &= 0 - \cos 0 + \cos 286\pi + \cos \frac{2004\pi}{7} + \cos \frac{2006\pi}{7} + \cos \frac{2008\pi}{7} \\ &= \cos \frac{2004\pi}{7} + \cos \frac{2006\pi}{7} + \cos \frac{2008\pi}{7} \\ &= \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7}. \end{aligned}$$

Now, we earlier noted that

$$\cos 0 + \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} + \cos \frac{8\pi}{7} + \cos \frac{10\pi}{7} + \cos \frac{12\pi}{7} = 0,$$

so

$$1 + \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} + \cos \frac{6\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{2\pi}{7} = 0.$$

Hence,

$$2 \left(\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} \right) = -1.$$

This means $2X = -1$, so $X = -0.5$. Hence,

$$|\lfloor 2008X \rfloor| = |\lfloor -1004 \rfloor| = |-1004| = 1004.$$

2. Jon wrote the n smallest perfect squares on one sheet of paper, and the n smallest triangular numbers on another (note that 0 is both square and triangular). Jon notices that there are the same number of triangular numbers on the first paper as there are squares on the second paper, but if n had been one smaller, this would not have been true. If $n < 2008$, let m be the greatest number Jon could have written on either paper. Find the remainder when m is divided by 2008.

Answer: 505

Credit: Problem by Mathew Crawford and Adam Hesterberg. Solution by Adam Hesterberg. Edits by Zach Abel.

Solution: We first seek numbers that are both squares and triangular numbers. If the x th square is the y th triangular number, then $2x^2 = y(y+1)$. Since y and $y+1$ have no common factors, one of them must be a square and the other must be twice a square, so $a^2 = 2b^2 \pm 1$. Note that $(a, b) = (1, 0)$ is a solution to this, and given (a, b) with $a^2 = 2b^2 \pm 1$, we also have

$$2(a+b)^2 = (a+2b)^2 \pm 1$$

This lets us generate infinitely many solutions to $a^2 = 2b^2 \pm 1$. These are in fact all solutions, because the process works in reverse: if (a, b) with $a, b > 1$ solves $a^2 = 2b^2 \pm 1$, then $a-b < b$ and $2b-a < a$, so

$$2(a-b)^2 = (2b-a)^2 \pm 1$$

is another solution with a and b smaller. Hence any solution can be reduced to one for which at least one of a and b is at most 1, but the only such solutions are $(a, b) = (1, 0)$ and $(a, b) = (1, 1)$. Hence the previous equation generates all possible solutions. The first few are:

a	b	x	y
1	0	0	0
1	1	1	1
3	2	6	8
7	5	35	49
17	12	204	288
41	29	1189	1681
99	70	6930	9800

Since $n < 2008$, there could not have been 7 square triangles on both papers, so the maximum number of them is 6, and the sixth is the 1189th square and the 1681st triangular number (where the 0th square and the 0th triangular number are 0). The greatest value of n less than 2008 for which this is not on both papers is 1681, so the greatest number Jon could have written is

$$m = 1681^2 = 2825761 = 1407 \cdot 2008 + 505.$$

3. Let $\phi = \frac{1+\sqrt{5}}{2}$ be the positive root of $x^2 = x + 1$. Define a function $f : \mathbb{N} \rightarrow \mathbb{N}$ by

$$f(0) = 1$$

$$f(2x) = \lfloor \phi f(x) \rfloor$$

$$f(2x + 1) = f(2x) + f(x).$$

Find the remainder when $f(2007)$ is divided by 2008.

Answer: 864

Credit: Problem and solution by Adam Hesterberg. Edited by Mathew Crawford and Zach Abel.

Solution: We claim that if $x = 2^{x_n} + 2^{x_{n-1}} + \dots + 2^{x_0}$ for integers $x_n > \dots > x_0$, then

$$f(x) = 1 + F_{x_0+2} + F_{x_1+3} + \dots + F_{x_n+n+2} \quad (\text{where } F_1 = F_2 = 1).$$

Then

$$\begin{aligned} f(11111010111_2) &= 1 + F_2 + F_4 + F_6 + F_9 + F_{12} + F_{14} + F_{16} + F_{18} + F_{20} \\ &= 1 + 1 + 3 + 8 + 34 + 144 + 377 + 987 + 2584 + 6765 \\ &= 10904 = 5 \cdot 2008 + 864. \end{aligned}$$

Proof: We induct on x . If $x = 1 = 2^0$, then $f(x) = 2 = 1 + F_2$.

If $x = 2^{x_n} + 2^{x_{n-1}} + \dots + 2^{x_0}$ is even, let $x = 2k$. Then

$$\begin{aligned} f(x) &= \lfloor \phi f(k) \rfloor \\ &= \lfloor \phi (1 + F_{x_0+1} + F_{x_1+2} + \dots + F_{x_n+n+1}) \rfloor \\ &= \left\lfloor \phi + \frac{\phi}{\sqrt{5}} (\phi^{x_0+1} - \phi^{-1-x_0} + \dots + \phi^{x_n+n+1} - \phi^{-1-n-x_n}) \right\rfloor \\ &= \left\lfloor \phi + \frac{1}{\sqrt{5}} (\phi^{x_0+2} - \phi^{0-x_0} + \dots + \phi^{x_n+n+2} - \phi^{-n-x_n}) \right\rfloor \\ &= \left\lfloor \phi + F_{x_0+2} + \dots + F_{x_n+n+2} + \frac{1}{\sqrt{5}} (-\phi^{-x_0-2} - \phi^{0-x_0} - \dots - \phi^{-x_n-n} - \phi^{2-n-x_n}) \right\rfloor \\ &= \left\lfloor 1 + F_{x_0+2} + \dots + F_{x_n+n+2} + \frac{1}{\phi} + \frac{1}{\sqrt{5}} (-\phi^{-x_0-2} - \phi^{0-x_0} - \dots - \phi^{-x_n-n} - \phi^{2-n-x_n}) \right\rfloor. \end{aligned}$$

But

$$\begin{aligned} 1 &> \frac{1}{\phi} + \frac{1}{\sqrt{5}} (-\phi^{-x_0-2} - \phi^{0-x_0} - \dots - \phi^{-x_n-n} - \phi^{2-n-x_n}) \\ &> \frac{1}{\phi} + \frac{1}{\sqrt{5}} (-\phi^{-3} - \phi^{-4} - \dots - \phi^{-1} - \phi^{-2} - \dots) \\ &= \frac{1}{\phi} - \phi^{-3} > 0, \end{aligned}$$

so

$$\begin{aligned} f(x) &= \left\lfloor 1 + F_{x_0+2} + \dots + F_{x_n+n+2} + \frac{1}{\phi} + \frac{1}{\sqrt{5}} (-\phi^{-x_0-2} - \phi^{0-x_0} - \dots - \phi^{-x_n-n} - \phi^{2-n-x_n}) \right\rfloor \\ &= \left\lfloor 1 + F_{x_0+2} + \dots + F_{x_n+n+2} + \frac{1}{\phi} \right\rfloor, \end{aligned}$$

as desired.

If $x = 2^{x_n} + 2^{x_{n-1}} + \cdots + 2^{x_1} + 2^0$ is odd, let $x = 2k + 1$. Then

$$\begin{aligned} f(x) &= f(2k) + f(k) \\ &= (1 + F_{x_1+2} + \cdots + F_{x_n+n+1}) + (1 + F_{x_1+1} + \cdots + F_{x_n+n}) \\ &= (2 + F_{x_1+3} + \cdots + F_{x_n+n+2}) \\ &= 1 + F_2 + F_{x_1+3} + \cdots + F_{x_n+n+2}, \end{aligned}$$

as desired.

4. If m is a positive integer, let S_m be the set of rational numbers in reduced form with denominator at most m . Let $f(m)$ be the sum of the numerator and denominator of the element of S_m closest to e (Euler's constant). Given that $f(2007) = 3722$, find the remainder when $f(1000)$ is divided by 2008.

Answer: 1993

Credit: Problem and solution by Adam Hesterberg. Edited by Zach Abel and Mathew Crawford.

Solution: We use continued fraction expansions

$$a = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

Definition: Call a rational number *Eulerian* (because he doesn't yet have enough named after him) if it is closer to $e = [e_0; e_1, \dots]$ than any rational number with smaller denominator.

Lemma: (1) If a rational number a is Eulerian, then a is expressible as $[a_0; a_1, \dots, a_n]$ where $a_0 = e_0$, $a_1 = e_1$, \dots , and $0 < a_n \leq e_n$.

(2) If $a_0 = e_0, \dots, a_n = e_n$, then a is Eulerian. (One can be more specific— see, for instance,

<http://mathworld.wolfram.com/ContinuedFraction.html>

– but more generality is unnecessary for this problem).

Let $g(m)$ be the element of S_m closest to e . By considering $e \approx 2.71828$ and using $\frac{1}{1+x} \approx 1 - x + x^2$, we find $\frac{2722}{1000} = 2.722 > e$, $\frac{2721}{1001} \approx 2.7182817 < e$, and $\frac{2720}{1002} \approx 2.714 < e$. Given that $f(2007) = 3722$, we must therefore have $g(2007) = \frac{2721}{1001}$.

Consider the continued fraction expansion of $\frac{2721}{1001}$,

$$[2; 1, 2, 1, 1, 4, 1, 1, 6, 2] = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1].$$

Since $\frac{2721}{1001} < e$ and

$$[2; 1, 2, 1, 1, 4, 1, 1, 6, 2 + x, y, \dots] < [2; 1, 2, 1, 1, 4, 1, 1, 6, 2],$$

the first few terms in the continued fraction expansion of e must be $[2; 1, 2, 1, 1, 4, 1, 1, 6, 1, x]$. When $x = 1$, we get the approximation $\frac{2721}{1001}$. When $x > 1$, we get a larger denominator. The next smaller denominator must therefore come from an Eulerian number with fewer terms in its continued fraction expansion. The nearest candidate is $[2; 1, 2, 1, 1, 4, 1, 1, 6, 1] = \frac{1457}{536}$. From part (1) of the lemma above, there are no other Eulerian numbers with denominator between 546 and 1000. Hence $g(1000) = \frac{1457}{536}$, so $f(1000) = 1993$.

5. Three circles with centers V_0, V_1, V_2 and radii 33, 30, 25 respectively are mutually externally tangent: P_i is the tangency point between circles V_{i+1} and V_{i+2} , where indices are taken modulo 3. For $i = 0, 1, 2$, line $P_{i+1}P_{i+2}$ intersects circle V_{i+1} at P_{i+2} and Q_i , and the same line intersects circle V_{i+2} at P_{i+1} and R_i . If Q_0R_1 intersects Q_2R_0 at X , then the distance from X to line R_1Q_2 can be expressed as $\frac{a\sqrt{b}}{c}$, where the integer b is not divisible by the square of any prime, and positive integers a and c are relatively prime. Find the value of $b + c$.

Answer: 114

Credit: Problem and solution by Zachary Abel.

Solution: Let $\ell_i = R_{i+1}Q_{i+2}$, and set $W_i = \ell_{i+1} \cap \ell_{i+2}$. Note that $P_0V_1P_2$ and $P_2V_0R_1$ are isosceles and share a base angle at P_2 , so they are similar. This means $\angle P_0V_1P_2 = \angle P_2V_0R_1$, so V_1V_2 and W_1W_2 are parallel. As the same argument holds for the other two sides of triangle $V_0V_1V_2$, we've found that triangle $V_0V_1V_2$ is the medial triangle of $W_0W_1W_2$, i.e. that V_i is the midpoint of $W_{i+1}W_{i+2}$ for each i .

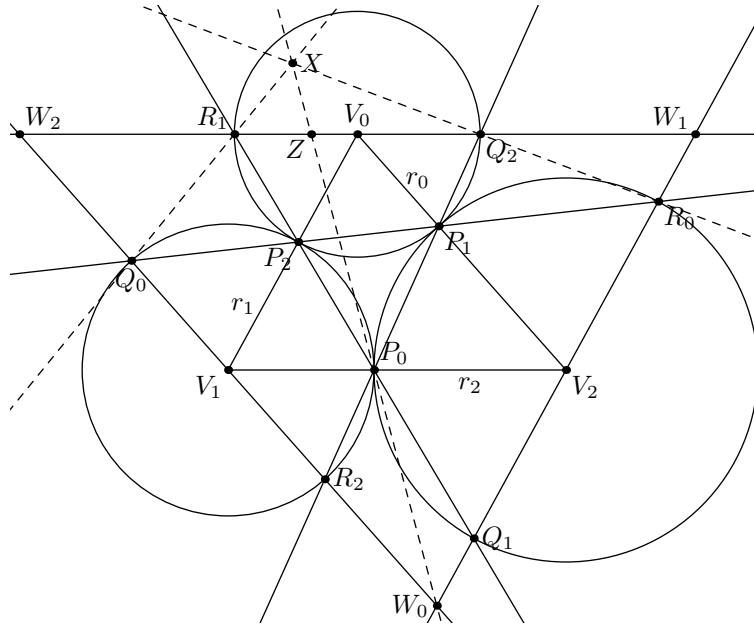


Figure 11: Problem 6-5.

The key observation is that X lies on line P_0W_0 . We'll prove this by Menelaus's theorem. Let r_i be the radius of circle V_i , and let P_0W_0 intersect ℓ_0 at Z . Suppose Q_0R_1 intersects W_0P_0 at X' . By Menelaus's theorem on triangle W_0W_2Z ,

$$\frac{W_0X'}{ZX'} = \frac{W_0Q_0}{Q_0W_2} \cdot \frac{W_2R_1}{R_1Z}.$$

Note that $W_0Q_0 = W_0V_1 + r_1 = V_0V_2 + r_1 = r_0 + r_1 + r_2$; the lengths $Q_0W_2 = r_0 + r_2 - r_1$ and $W_2R_1 = r_1 + r_2 - r_0$ may be calculated similarly. Finally, $R_1Z = W_2Z - W_2R_1 = 2V_1P_0 - W_2R_1 = r_0 + r_1 - r_2$, so

$$\frac{W_0X'}{ZX'} = \frac{r_0 + r_1 + r_2}{r_0 + r_2 - r_1} \cdot \frac{r_1 + r_2 - r_0}{r_0 + r_1 - r_2} = t. \tag{5}$$

If the same argument is carried out for $W_0P_0 \cap R_0Q_2 = X''$ and triangle W_0W_1Z , the same value

$$\frac{W_0X''}{ZX''} = \frac{r_0 + r_1 + r_2}{r_0 + r_1 - r_2} \frac{r_1 + r_2 - r_0}{r_0 + r_2 - r_1} = t.$$

will be found. Thus, $X' = X'' = X$, as desired.

Let h be the length of the altitude from V_0 to V_1V_2 , and let A be the length of the altitude from X to R_1Q_2 . The above calculation guarantees that

$$\frac{W_0X}{ZX} = t = \frac{2h + A}{A},$$

and solving gives $A = \frac{2h}{t-1}$. With the given values $\{r_0, r_1, r_2\} = \{33, 30, 25\}$, the necessary values may be calculated. Equation (5) gives $t = \frac{88 \cdot 22}{38 \cdot 28} = \frac{242}{133}$. The area of $\triangle V_0V_1V_2$ may be calculated with Heron's formula: $[V_0V_1V_2] = \sqrt{88 \cdot 33 \cdot 30 \cdot 25} = 660\sqrt{5}$, and from here, we have $h = \frac{2[V_0V_1V_2]}{V_1V_2} = 24\sqrt{5}$. Finally, $A = \frac{2h}{t-1} = \frac{6384\sqrt{5}}{109}$. The desired answer is $5 + 109 = 114$.