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The following problems are old ARML Power Questions. The fifteen members of an ARML team are given one hour to write rigorous solutions to the Power Question. Note that over the years, the Power Question has become more in-depth and more difficult.

It is assumed that students new to ARML will have great difficulty with the following problems. An inability to write proofs should not discourage your students. Learning to write proofs is often a long and difficult process even for undergraduate math students. One of the goals of Alabama ARML is simply to help each student improve their solution-writing skills gradually throughout the year so that every student selected for the ARML team has the opportunity to contribute to at least one part of the Power Question.

## 1977 Atlantic Regions Mathematics League

**Power Question**

1. Let $S$ be an integer. Prove that $S$ cannot be expressed as the sum of two or more consecutive positive odd integers if
   (a) $S$ is a prime,
   (b) $S$ is twice an odd integer.
2. Prove that if an integer greater than 1 is neither a prime nor twice an odd integer, then it can be expressed as the sum of two or more consecutive positive odd integers.
3. Classify all integers which have precisely one representation as a sum of two or more consecutive positive odd integers.

## 1980 Atlantic Regions Mathematics League

**Power Question**

1. Formulate and prove a necessary and sufficient condition for 3 given real numbers to be the lengths of the sides of a triangle.
2. (a) Prove that there exist infinitely many "integer-sided" non-isosceles triangles such that the length of one side is a square, the length of another side is a power of 2, and the length of teh third side is a prime.
   (b) Prove or disprove that, for every prime $p$, there exists at least one "integer-sided" non-isosceles triangle such that the length of one side is a square, the length of another side is a power of 2, and the length of the third side is $p$.
3. Let $T(n)$ equal the number of incongruent triangles whose side-lengths are positive integers less than or equal to the positive integer $n$.
   (a) Verify that $T(1) = 1$ and $T(2) = 3$.

(b) Find a formula for $T(n)$ as a polynomial in $n$ with rational coefficients. [Note: The formula for $T(n)$ will be different for $n$ even and for $n$ odd.]

OPTIONAL: Investigate the function $D(n)$ equal to the number of non-similar triangles whose side-lengths are positive integers less than or equal to $n$ (where $n$ is a positive integer). Possible objectives of this investigation might be a study of the behavior of $D(n)/n^3$ and/or $T(n)/D(n)$ [or just the calculation of $D(9)$].

## 1991 American Regions Mathematics League

**Power Question – "Nice Angles" and Polygons**

DEFINITION 1: We call angle $A$ "nice" if *both* $\sin A$ and $\cos A$ are rational.
DEFINITION 2: We call a convex polygon "nice" if all of its interior angles are "nice".

I. Prove each of the following:

1a. If angle $A$ is nice, its supplement will be nice.

1b. If angle $A$ is nice, $\frac{1}{2}A$ need not be nice, but $2A$ will be nice.

1c. The set of nice angles is closed under addition.

2a. [Note: A Pythagorean triangle is a right triangle with integer sides.] Every acute nice angle is the angle of some Pythagorean triangle.

2b. If angle $A$ is an acute nice angle and $\cos A = b/c$, where $b$ and $c$ are relatively prime positive integers, then $c$ will be odd.

2c. There is no smallest positive nice angle.

II. 1. Prove that if the sides of a triangle are rational, and one angle is nice, then the other angles will be nice. [Be sure you satisfy both parts of Definition 1.]

2a. Find the sides of an *acute* nice triangle whose sides are integers and whose perimeter is less than 20. [Be sure that this triangle is both acute and nice.]

2b. Prove that if the sides of a nice triangle are integers, its area will be an *even integer*. [Hint: One approach would be to first show that its perimeter must be even.]

III. A convex quadrilateral has the following properties:

1. Its sides are integers whose product is a square.
2. It can be inscribed in a circle, and can be circumscribed about (another) circle.

Prove that this quadrilateral is nice.

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# 1998 American Regions Mathematics League

**Power Question – Mediations on Partitions**

For each of the following problems write complete solutions. Just each answer. Angles are in degrees.

1. Let positive integers $A$, $B$, and $C$ be the angles of a triangle such that $A \leq B \leq C$.
   a) Determine all the values that each of $A$, $B$, and $C$ can take on.
   b) Compute the number of ordered triples $(A, B, C)$ in which $B = 70°$.

2. In convex pentagon $ABCDE$, $m\angle A < m\angle B < m\angle C < m\angle D < m\angle E$. Let $T = m\angle C + m\angle D$. If $m\angle A : m\angle B : m\angle C : m\angle D : m\angle E = 1 : 2 : x : y : 5$, determine the range of values of $T$.

3. Let $a$, $b$, and $c$ be positive integers such that $a < 3b$, $b > 4c$, and $a + b + c = 200$.
   a) Determine the largest value that $c$ can take on.
   b) Determine the smallest value that $b$ can take on.
   c) Determine the number of ordered triples $(a, b, c)$ in which $c = 11$.

4. Let $a, b$, and $c$ be positive integers. If $a + b + c = 85$, $c > 3a$, $2b > c$, and $5a > 3b$, prove algebraically that there is a unique solution $(a, b, c)$ to this system.

5. A unit square is divided into 4 rectangles of positive area by two cuts parallel to the sides of the square. Let $a_1 \leq a_2 \leq a_3 \leq a_4$ be the area of the four parts in non-decreasing order. For each $i = 1, \ldots, 4$, determine with proof the range of values for $a_i$.

6. A unit cube is divided into 8 parallelepipeds of positive volume by three cuts parallel to the faces of the cube. Let $v_1 \leq v_2 \leq \cdots \leq v_8$ be the volumes of the eight parts in non-decreasing order. Determine with proof the range of values for $v_4$ and $v_5$.

7. Let $n$ be a positive integer. Allie and Bob play a game constructing a partition $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$. Allie wins if there is an odd number of terms in the partition, i.e., if $k$ is odd; Bob wins otherwise. Allie begins by choosing a number $a_1$ between 1 and $n - 1$ inclusive. Bob then chooses a number $a_2$ between $a_1$ and 1 inclusive such that $a_1 + a_2 \leq n$. Allie then chooses an $a_3$ between $a_2$ and 1 inclusive such that $a_1 + a_2 + a_3 \leq n$, and so on, with the game ending when the partition is complete. Determine with proof all $n > 1$ for which Bob has a winning strategy.

8. Allie and Bob play a game similar to the one in #7 except that the inequality $a_i \geq a_{i+1}$ is replaced by $2a_i \geq a_{i+1}$. Prove that Bob has a winning strategy if and only if $n$ is a Fibonacci number. (You may assume the following: each positive integer $n$ can be uniquely represented as a decreasing sum of non-adjacent Fibonacci numbers, i.e., $32 = 21 + 8 + 3$.)