This document is a sample of the MIST Academy Advanced Problem Solving curriculum. Your child/student may find some of these problems easy and others very difficult. Students are working with the right curriculum when they get some problems and struggle with others.

Algebra

MIST Academy curriculum covers all the topics in a standard pre-algebra and algebra I course. It also includes a host of problem solving techniques and a particular focus on the art of translating words into math. Students who learn algebra as fluently as a language are at a great advantage when it comes to applying their knowledge.

1. Suppose that x^3+px^2+qx+r is a cubic with a double root at a and another root at b, where a and b are real numbers. If $p=-6$ and $q=9$, what is r? *(iTest)*

2. If $x>1$ and if
$$(\log_x 128)(\log_{128} 16) = y,$$
determine the value of $\log_{2x} 256$, expressing your answer as a quotient of two linear polynomials in y whose leading coefficients are relatively prime positive integers. *(ARML)*

3. In Pascal's Triangle, each entry is the sum of the two entries above it. In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3:4:5$? *(AIME)*

4. Find the product of the nonreal roots of the equation
$$(x^2-3x)^2+5(x^2-3x)+6=0.$$
(iTest)

5. Find the sum of the solutions to the equation
$$\sqrt[4]{x+27}+\sqrt[4]{55-x}=4.$$
(Alabama ARML TST)

6. The roots of $x^4-Kx^3+Kx^2+Lx+M=0$ are $a,b,c,$ and d. If K, L, and M are real numbers, compute the minimum value of the sum $a^2+b^2+c^2+d^2$. *(ARML)*

7. Find the greatest natural number possessing the property that each of its digits except the first and last one is less than the arithmetic mean of the two neighboring digits. *(Leningrad Mathematical Olympiad)*

8. Let S be the sum of all x such that $1\le x\le 99$ and
$$\{x^2\}=\{x\}^2.$$
Compute $\lfloor S\rfloor$. *(iTest)*

Building Bridges

Learning to solve more and more difficult math problems involves overcoming precisely what makes them more difficult. Students in the Advanced Problem Solving course are expected to already have a grasp on all aspects of a pre-calculus high school math curriculum – at the highest levels ordinarily tested. So, the math that challenge these students require a broader understanding of problem solving techniques, greater exploration, and the ability to apply several areas of mathematics or techniques to a single problem.

1. Compute the area of the solution set of
$$\lfloor x \rfloor \cdot \lfloor y \rfloor = 2000.$$
(ARML)

2. Let x be the length of one side of a triangle and let y be the height to that side. If $x+y = 418$, find the maximum possible *integral value* of the area of the triangle. *(iTest)*

3. A positive integer n between 1 and $N = 2007^{2007}$ inclusive is selected at random. If a and b are natural numbers such that a/b is the probability that N and $n^3 - 36n$ are relatively prime, find the value of $a + b$. *(iTest)*

4. Find the real solution (x, y) to the system of equations
$$x^3 - 3xy^2 = -610,$$
$$3x^2y - y^3 = 182.$$
(Alabama ARML TST)

5. In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned 1/2 point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of her/his points against the other nine of the ten). What was the total number of players in the tournament? *(AIME)*

6. Find with proof all integers n such that $x^4 + n$ can be factored into two distinct trinomial factors with integer coefficients. *(ARML)*

7. In trapezoid $ABCD$, with $AB < DC$, the sum of the areas of regions I and II is 1996. If the lengths of bases $\overline{AB}$ and $\overline{CD}$ are integers and the distance between them is an integer, compute the minimum area of $ABCD$. *(ARML)*

Combinatorics

Combinatorics is the mathematician's fancy word for the study of counting techniques. In the Advanced Problem Solving class, we completely rebuild our foundation of combinatorics and use a greater depth of understanding of counting techniques to solve highly challenging problems.

1. How many *odd* four-digit integers have the property that their digits, read left to right, are in strictly decreasing order? *(iTest)*

2. When Jon Stewart walks up stairs he takes one or two steps at a time. His stepping sequence is not necessarily regular. He might step up one step, then two, then two again, then one, then one, and then two in order to climb up a total of 9 steps. In how many ways can Jon walk up a 14 step stairwell? *(Alabama ARML TST)*

3. Let $(a_1, a_2, a_3, \ldots, a_{12})$ be a permutation of $(1, 2, 3, \ldots, 12)$ for which

$$a_1 > a_2 > a_3 > a_4 > a_5 > a_6 \text{ and } a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}.$$

   An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations. *(AIME)*

4. Ted's favorite number is equal to

$$1 \cdot \binom{2007}{1} + 2 \cdot \binom{2007}{2} + 3 \cdot \binom{2007}{3} + \cdots + 2007 \cdot \binom{2007}{2007}.$$

   Find the remainder when Ted's favorite number is divided by 25. *(iTest)*

5. A collection of 8 cubes consists of one cube with edge-length k for each integer k, $1 \le k \le 8$. A tower is to be built using all 8 cubes according to the rules:

   - Any cube may be the bottom cube in the tower.
   - The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

   Let T be the number of different towers that can be constructed. What is the remainder when T is divided by 1000? *(AIME)*

6. The sequence of digits

$$123456789101112131415161718192021\ldots$$

   is obtained by writing the positive integers in order. If the 10^nth digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100^{th} digit enters the sequence in the placement of the two-digit integer 55. Find the value of $f(2007)$. *(iTest)*

Complex Numbers

Even most advanced high school curriculae do not do justice to the study of complex numbers. This area of mathematics is particularly important for the study of a great deal of higher math and also physics.

1. Suppose that n is a positive integer and that $z + \dfrac{1}{z} = 2\cos\theta$, where $0 < \theta < \pi$. What is the value of $z^n + \dfrac{1}{z^n}$? *(AHSME)*

2. Evaluate and simplify: $\left(\sqrt{3}+i\right)^{12} + \left(\sqrt{3}-i\right)^{12}$.

3. Show that there is exactly one complex number z such that

$$|z-3| = |z+2i| = |z+1-4i|.$$

4. Let $\omega^3 = 1$, where $\omega \neq 1$, find the value of

$$(1-\omega+\omega^2)(1+\omega-\omega^2).$$

5. Consider the region A in the complex plane that consists of all points z such that both $z/40$ and $40/\overline{z}$ have real and imaginary parts between 0 and 1 inclusive. What is the integer that is nearest the area of A? (If $z = x + iy$ with x and y real, then $\overline{z} = x - iy$ is the conjugate of z.) *(AIME)*

6. Express i^i as a real number without imaginary parts.

7. If $\cos\theta = \frac{1}{5}$, then evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$.

Games and Puzzles

The following is a samples of games and puzzles that may be used with Advanced Problem Solving students in the MIST Academy classroom. Results of these games will be generalized.

1. A pawn is placed in the central square of a 11×11 chessboard. Two players move the pawn in succession to any other square, but each move (beginning with the second) must be longer than the previous one. The player who cannot make such a move loses. Who wins in an errorless game? *(Leningrad Mathematical Olympiad)*

2. Aliquot part is another name for a proper divisor, i.e. any divisor of a given number other than the number itself. A prime number has only one aliquot part – the number 1. 1, 2, 3, 4, 6 are all aliquot parts of 12. The number 1 does not have aliquot parts. In the Aliquot game, players take turns subtracting an aliquot part of the number left by their opponent. The winner is the last player able to perform such a subtraction. The loser is the player left with a number that has no aliquot parts – 1. Thus the objective of the game is to leave your opponent without a move.

3. Alfred and Bonnie play a game in which they take turns tossing a fair coin. The winner of a game is the first person to obtain a head. Alfred and Bonnie play this game several times with the stipulation that the loser of a game goes first in the next game. Suppose that Alfred goes first in the first game, and that the probability that he wins the sixth game is m/n, where m, and n, are relatively prime positive integers. What are the last three digits of $m + n$? *(AIME)*

4. Two players play the following game with a fair coin. Player 1 chooses (and announces) a triplet (HHH, HHT, HTH, HTT, THH, THT, TTH, or TTT) that might result from three successive tosses of the coin. Player 2 then chooses a different triplet. The players toss the coin until one of the two named triplets appears. The triplets may appear in any three consecutive tosses: (1st, 2nd, 3rd), (2nd, 3rd, 4th), and so on. The winner is the player whose triplet appears first.

    1. What is the optimal strategy for each player? With best play, who is most likely to win?
    2. Suppose the triplets were chosen in secret? What then would be the optimal strategy?
    3. What would be the optimal strategy against a randomly selected triplet?

5. A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking **either**

    (a) one bean from a heap, provided at least two beans are left behind in that heap,
    or

    (b) a complete heap of two or three beans.

    The player who takes the last heap wins. Two win the game, do you want to move first or second? Give a winning strategy. *(Putnam)*

Geometry

In addition to the kinds of problems listed below, Advanced Problem Solving students will relearn geometry from the ground up in order to understand the underpinnings of Euclidean geometry.

1. Four equilateral triangles are drawn such that each one shares a different side with a square of side length 10. None of the areas of the triangles overlap with the area of the square. The four vertices of the triangles that aren't vertices of the square are connected to form a larger square. Find the area of this larger square. *(Alabama ARML TST)*

2. An equilateral triangle of side 12 is inscribed in a circle. Diameter $\overline{AB}$ is parallel to one side of the triangle and intersects the other sides at points C and D, with point C closer to A. If $AC = p\sqrt{3} - q$, compute the ordered pair of rational numbers (p, q). *(ARML)*

3. In an isosceles trapezoid, the length of each leg is 3, the length of each diagonal is 7, and the length of the longer base is 8. Find the length of the shorter base. *(ARML)*

4. In concave hexagon $ABCDEF$, $\angle A = \angle B = \angle C = 90°$, $\angle D = 100°$, and $\angle F = 80°$. Also, $CD = FA$, $AB = 7$, $BC = 10$, and $EF + DE = 12$. Compute the area of the hexagon. *(Alabama ARML TST)*

5. An equilateral triangle of side 12 is inscribed in a circle. Diameter $\overline{AB}$ is parallel to one side of the triangle and intersects the other sides at points C and D, with point C closer to A. If $AC = p\sqrt{3} - q$, compute the ordered pair of rational numbers (p, q). *(ARML)*

6. The union of the infinity of line segments $\overline{P_1P_2}, \overline{P_2P_3}, \ldots, \overline{P_nP_{n+1}}$ is called the *zig-zag line* U. When drawn in the rectangular coordinate plane, each point P_n of U is on the x-axis if and only if n is odd, and is on the graph of $y = (x-2)^2$ if and only if n is even. Furthermore, the degree-measure of the angle between any two segments of U which share a common endpoint is 60. If P_1 is at the origin, find the length of U from P_1 to the point P_k whose coordinates are $(2, 0)$, if $m\angle P_2P_1P_3 = 60$. *(ARML)*

7. A circle with radius one is centered at the origin. Four points, A, B, C, and D, are chosen on the circle so that AC and BD intersect at $(\frac{1}{2}, 0)$ and $AC \perp BD$. Find the maximum possible area of quadrilateral $ABCD$. *(Mandelbrot)*

Number Theory

Number theory, the mathematics of integers is one of the most unfortunately under-covered areas of mathematics, especially at the middle school level. In a world in which computing is largely based on logic and number theory, MIST Academy promotes number theory as a substantial part of its curriculum.

1. Determine all positive primes p such that $p^{1994} + p^{1995}$ is a perfect square.

2. There exist positive integers A, B, C, and D with no common factor greater than 1, such that
$$A \log_{1200} 2 + B \log_{1200} 3 + C \log_{1200} 5 = D.$$
Find $A + B + C + D$. *(Alabama ARML TST)*

3. Let a_n equal $6^n + 8^n$. Determine the remainder upon dividing a_{83} by 49. *(AIME)*

4. What is the smallest positive integer k such that the number $\binom{2k}{k}$ ends in two zeros? *(iTest)*

5. Let S be the sum of all numbers of the form a/b, where a and b are relatively prime positive divisors of 1000. What is the greatest integer that does not exceed $S/10$? *(AIME)*

6. Let $\phi(n)$ be the number of positive integers $k < n$ which are relatively prime to n. For how many distinct values of n is $\phi(n)$ equal to 12? *(iTest)*

7. Assume that $a, b, c,$ and d are positive integers such that $a^5 = b^4$, $c^3 = d^2$, and $c - a = 19$. Determine $d - b$. *(AIME)*

8. There is one natural number with exactly 6 positive divisors, the sum of whose reciprocals is 2. Find that natural number. *(Alabama ARML TST)*

9. Let $[r, s]$ denote the least common multiple of positive integers r and s. Find the number of ordered triples (a, b, c) of positive integers for which $[a, b] = 1000$, $[b, c] = 2000$, and $[c, a] = 2000$. *(AIME)*

10. The integer 5^{2006} has 1403 digits, and 1 is its first digit (farthest to the left). For how many integers $0 \le k \le 2005$ does 5^k begin with the digit 1? *(Alabama ARML TST)*

11. Find the largest positive integer that is equal to the cube of the sum of its digits. *(iTest)*

12. Yoda begins writing the positive integers starting from 1 and continuing consecutively as he writes. When he stops, he realizes that there is no set of 5 composite integers among the ones he wrote such that each pair of those 5 is relatively prime. What's the largest possible number Yoda could have stopped on? *(Alabama ARML TST)*

Probability

1. River draws four cards from a standard 52 card deck of playing cards. Exactly 3 of them are 2's. Find the probability River drew exactly one spade and one club from the deck. *(Alabama ARML TST)*

2. Three distinct points are, independently, randomly selected on a circle. Find the probability that the triangle having these three points as vertices is an obtuse triangle. *(ARML)*

3. When a certain biased coin is flipped five times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let $\frac{i}{j}$, in lowest terms, be the probability that the coin comes up heads in exactly 3 out of 5 flips. Find $i+j$. *(AIME)*

4. Yeechi has a deck of cards consisting of the 2 through 5 of hearts and the 2 through 5 of spades. She deals two cards at random to each of four players. What is the probability that no player receives a pair? *(Mandelbrot)*

5. A fair die is rolled four times. The probability that each of the final three rolls is at least as large as the roll preceding it may be expressed in the form m/n, where m and n are relatively prime positive integers. Find $m+n$. *(AIME)*

6. Let b be a real number randomly selected from the interval $[-17, 17]$. Then, m and n are two relatively prime positive integers such that m/n is the probability that the equation

$$x^4 + 25b^2 = (4b^2 - 10b)x^2$$

has *at least* two distinct real solutions. Find the value of $m+n$. *(iTest)*

7. A fair coin is to be tossed 10 times. Let i/j, in lowest terms, be the probability that heads never occur on consecutive tosses. Find $i+j$. *(AIME)*

8. Ying lives on Strangeland, a tiny planet with 4 little cities that are each 100 miles apart from each other. One day, Ying begins driving from her home city of Viavesta to the city of Havennew, which takes her about an hour. When she gets to Havennew, she decides she wants to go straight to another city on Strangeland, so she randomly chooses one of the other three cities (possibly Viavesta), and starts driving there. Ying drives like this for most of the day, making 8 total trips between cities on Strangeland, choosing randomly where to drive to next from each stop. She then stops at her final city of destination, digs a hole, and buries her car.

   Let p be the probability Ying buried her car in Viavesta and let q be the probability she buried it in Havennew. Find the value of $p+q$. *(Alabama ARML TST)*

9. Two three-letter strings, *aaa* and *bbb*, are transmitted electronically. Each string is sent letter by letter. Due to faulty equipment, each of the six letters has a 1/3 chance of being received

incorrectly, as an a when it should have been a b, or as a b when it should be an a. However, whether a given letter is received correctly or incorrectly is independent of the reception of any other letter. let S_a be the three-letter string received when aaa is transmitted and let S_b be the three-letter string received when bbb is transmitted. Let p be the probabililty that S_a comes before S_b in alphabetical order. When p is written as a fraction in lowest terms, what is its numerator? *(AIME)*

10. A drawer contains a mixture of red socks and blue socks, at most 1991 in all. It so happens that, when two socks are selected randomly without replacement, there is a probability of exactly $\frac{1}{2}$ that both are red or both are blue. What is the largest possible number of red socks in the drawer that is consistent with this data? *(AIME)*

11. During a movie shoot, a stuntman jumps out of a plane and parachutes to safety within a 100 foot by 100 foot square field, which is entirely surrounded by a wooden fence. There is a flag pole in the middle of the square field. Assuming the stuntman is equally likely to land on any point in the field, the probability that he lands closer to the fence than to the flag pole can be written in simplest terms as

$$\frac{a - b\sqrt{c}}{d},$$

where all four variables are positive integers, c is a multiple of no perfect square greater than 1, a is coprime with d, and b is coprime with d. Find the value of $a + b + c + d$. *(iTest)*

Sequences & Series

Problems involving sequences and series can challenge a student to apply master of arithmetic, pattern hunting, algebra, number theory, and sometimes other areas of mathematics. These problems often frustrate even talented solvers, but can be the most enjoyable at the same time.

1. Evaluate $\sum_{i=1}^{6}\sum_{j=1}^{8} ij$.

2. Define $p = \sum_{k=1}^{\infty} \frac{1}{k^2}$ and $q = \sum_{k=1}^{\infty} \frac{1}{k^3}$. Find a way to write $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(j+k)^3}$ in terms of p and q. *(Mandelbrot)*

3. Consider the triangular array of numbers 0, 1, 2, 3, ... along the sides and interior numbers obtained by adding the two adjacent numbers in the previous row. Rows 1 through 6 are shown.

$$\begin{array}{ccccccccccc} & & & & & 0 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 2 & & 2 & & 2 & & & \\ & & 3 & & 4 & & 4 & & 3 & & \\ & 4 & & 7 & & 8 & & 7 & & 4 & \\ 5 & & 11 & & 15 & & 15 & & 11 & & 5 \end{array}$$

Let $f(n)$ denote the sum of the numbers in row n. What is the remainder when $f(100)$ is divided by 100? *(AHSME)*

4. One day Jason finishes his math homework early, and decides to take a jog through his neighborhood. While jogging, Jason trips over a leprechaun. After dusting himself off and apologizing to the odd little magical creature, Jason, thinking there is nothing unusual about the situation, starts jogging again. Immediately the leprechaun calls out, "hey, stupid, this is your only chance to win gold from a leprechaun!"

   Jason, while not particularly greedy, recognizes the value of gold. Thinking about his limited college savings, Jason approaches the leprechaun and asks about the opportunity. The leprechaun hands Jason a fair coin and tells him to flip it as many times as it takes to flip a head. For each tail Jason flips, the leprechaun promises one gold coin.

   If Jason flips a head right away, he wins nothing. If he first flips a tail, then a head, he wins one gold coin. If he's lucky and flips ten tails before the first head, he wins *ten gold coins*. What is the expected number of gold coins Jason wins at this game? *(iTest)*

5. If P is the product of n quantities in geometric progression, S their sum, and S' the sum of their reciprocals, then find P in terms of S, S', and n. *(AHSME)*

6. If the integer k is added to each of the numbers 36, 300, and 596, one obtains the squares of three consecutive terms of an arithmetic sequence. Find k. *(AIME)*

7. When Jon Stewart walks up stairs he takes one or two steps at a time. His stepping sequence is not necessarily regular. He might step up one step, then two, then two again, then one, then one, and then two in order to climb up a total of 9 steps. In how many ways can Jon walk up a 14 step stairwell? *(Alabama ARML TST)*

8. Let a, b, and c be positive real numbers such that a, b, c forms a harmonic sequence. Demonstrate that $a/(b+c)$, $b/(a+c)$, $c/(a+b)$ also forms a harmonic progression.

9. Evaluate $\displaystyle\prod_{n=1}^{13} \frac{n(n+2)}{(n+4)^2}$. *(Mandelbrot)*

10. Determine the value of the infinite product $2^{1/3} \cdot 4^{1/9} \cdot 8^{1/27} \cdot 16^{1/81} \cdots$. *(Mandelbrot)*

11. Find the sum of the infinite series:

$$3 + \frac{11}{4} + \frac{9}{4} + \cdots + \frac{n^2+2n+3}{2^n} + \cdots.$$

(Alabama ARML TST)

12. Assume that x_1, x_2, $\ldots$, x_7 are real numbers such that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1, \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12, \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123. \end{aligned}$$

Find the value of

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7.$$

(AIME)

13. Consider the sequence $(1, 2007)$. Inserting the difference between 1 and 2007 between them, we get the sequence $(1, 2006, 2007)$. Repeating the process of inserting differences between numbers, we get the sequence $(1, 2005, 2006, 1, 2007)$. A third iteration of this process results in $(1, 2004, 2005, 1, 2006, 2005, 1, 2006, 2007)$. A total of 2007 iterations produces a sequence with $2^{2007}+1$ terms. Find the number of times that 2004 appears among those $2^{2007}+1$ terms. *(iTest)*

Trigonometry

The MIST Academy Advanced Problem Solving class includes a great deal of trigonometry, including both the algebraic and geometric aspects of the subject, along with its connection to complex numbers.

1. Find the numerical value of
$$\frac{\sin 18^\circ \cos 12^\circ + \cos 162^\circ \cos 102^\circ}{\sin 22^\circ \cos 8^\circ + \cos 158^\circ \cos 98^\circ}.$$
(ARML)

2. Compute the number of degrees in the smallest positive angle x such that
$$8 \sin x \cos^5 x - 8 \sin^5 x \cos x = 1.$$
(ARML)

3. If $0^\circ < x < 180^\circ$ and $\cos x + \sin x = \frac{1}{2}$, then $\tan x = -\left(\frac{p+\sqrt{q}}{3}\right)$. Compute the ordered pair of integers (p, q). *(ARML)*

4. If x and y are acute angles such that $x + y = \pi/4$ and $\tan y = 1/6$, find the value of $\tan x$. *(iTest)*

5. Right triangle ABC, with $\angle C = 90^\circ$, lies in plane P with $a = 3$ and $b = 4$. Segments $\overline{AR}$, $\overline{BS}$, and $\overline{CT}$ are drawn perpendicular to, and on the same side of, plane P. If $AR = 7$, $BS = 6$, and $CT = 10$, compute the cosine of $\angle RTS$. *(ARML)*

6. Isosceles triangle AOB is inscribed in the parabola $y = x^2$. Vertex O is at the origin, and base $\overline{AB}$ is parallel to the x-axis. If $\tan \angle AOB = 3/4$, compute the y-coordinate of point B. *(ARML)*

7. Points A, B, C, and D lie equally spaced along a line in the order given with $AB = BC = CD = 1$. A point P is located so that $\sin(m\angle APC) = \frac{3}{5}$ and $\sin(m\angle BPD) = \frac{4}{5}$. Determine $\sin(2m\angle BPC)$. *(Mandelbrot)*

8. Let a and b be relatively prime positive integers such that a/b is the maximum possible value of
$$\sin^2 x_1 + \sin^2 x_2 + \sin^2 x_3 + \cdots + \sin^2 x_{2007},$$
where, for $1 \le i \le 2007$, x_i is a nonnegative real number, and
$$x_1 + x_2 + x_3 + \cdots + x_{2007} = \pi.$$
Find the value of $a + b$. *(iTest)*