1. Compute: $\sqrt{2000 \cdot 2007 \cdot 2008 \cdot 2015 + 784}$.

   Answer: 4030028

   Solution 1: There are certainly other [mostly longer] routes through the computations, but here we choose to let $x = 2000$:

$$\begin{aligned}\sqrt{2000 \cdot 2007 \cdot 2008 \cdot 2015 + 784} &= \sqrt{x(x+7)(x+8)(x+15)+784} \\ &= \sqrt{[x(x+15)][(x+7)(x+8)]+784} \\ &= \sqrt{(x^2+15x)(x^2+15x+56)+784} \\ &= \sqrt{[(x^2+15x+28)-28][(x^2+15x+28)+28]+784} \\ &= \sqrt{(x^2+15x+28)^2-28^2+784} \\ &= \sqrt{(x^2+15x+28)^2} = x^2+15x+28 \\ &= 4000000+30000+28 = 4030028.\end{aligned}$$

   Solution 2: We proceed as before, but we make the substitution $y = x^2 + 15x + 28$ to make the last few steps of algebra and arithmetic even more clear:

$$\begin{aligned}\sqrt{(x^2+15x)(x^2+15x+56)+784} &= \sqrt{[(x^2+15x+28)-28][(x^2+15x+28)+28]+784} \\ &= \sqrt{(y-28)(y+28)+784} \\ &= \sqrt{y^2-28^2+784} \\ &= \sqrt{y^2} = y = x^2+15x+28 \\ &= 4000000+30000+28 = 4030028.\end{aligned}$$

2. The points A, B, C, and D lie in that order on a line. Point E lies in a plane with A, B, C, and D such that $\angle BEC = 78^\circ$. Given that $\angle EBC > \angle ECB$, $\angle ABE = 4x + y$, and $\angle ECB = x + y$, compute the number of positive integer values that y can take on.

   Answer: 24

   Solution: Returning ARML students may note that this problem is very similar to the first individual problem from the 2007 ARML contest. This is intentional. Any first team member should correctly answer this problem nearly all the time (though we all suffer from the occasional careless error).

   An exterior angle of a triangle is equal to the sum of the other two angles of a triangle, so

$$4x + y = 78 + x + y \qquad \Rightarrow \qquad 3x = 78 \qquad \Rightarrow \qquad x = 26.$$

   The other given information is that $\angle EBC > \angle ECB$, so

$$180^\circ - (4x + y) > x + y \qquad \Rightarrow \qquad 76 - y > 26 + y \qquad \Rightarrow \qquad 25 > y.$$

   Hence, y can be integers from 1 to 24 inclusive.

3. Back in 1802, the town idiot of Boratsville, Thungar the Stenchified, bought some apples from a vendor at the market. He paid 12 cents for the apples, but when he saw how small

the apples were, he demanded two extra apples for free, and the vendor agreed. Ultimately, the price Thungar paid was one penny less per dozen apples than he would have paid had he not received the extra apples. How many apples did Thungar purchase (including the two extra)?

Answer: 18

Solution: We let x be the number of apples Thungar walked away with. He paid 12 cents for them, so his price per apples was $12/x$, and so the price per dozen was

$$12 \cdot \frac{12}{x} = \frac{144}{x}.$$

Had he not received two extra apples, he would have walked away with $x - 2$ apples, for a price per dozen of $144/(x - 2)$. This price would have been a penny higher, so

$$\frac{144}{x-2} - \frac{144}{x} = 1 \qquad \Rightarrow \qquad 144x - 144(x-2) = x(x-2).$$

The left-hand side of our new quadratic equation simplifies, and we can solve the equation easily:

$$288 = x^2 - 2x \qquad \Leftrightarrow \qquad x^2 - 2x - 288 = (x-18)(x+16) = 0.$$

Of course, $x > 0$, so $x = 18$ meaning Thungar purchased 18 apples.

4. Find the smallest positive integer N such that the product $19999N$ ends in the four digits 2007.

Answer: 7993

Solution 1: After a little experimentation, we note that the last four digits of the multiples of 19999 "tick down" by 1 each time N goes up by 1:

$$\begin{aligned} 19999 \cdot 1 &= 1\underline{9999}, \\ 19999 \cdot 2 &= 3\underline{9999}, \\ 19999 \cdot 3 &= 5\underline{9999}, \\ &\vdots \\ 19999 \cdot 7993 &= 15985\underline{2007}. \end{aligned}$$

Solution 2: The last four digits of a positive integer represent its modulo 10000 residue, and we note that $19999 \equiv -1 \pmod{10000}$. So,

$$19999N \equiv -1 \cdot N \equiv -N \equiv 2007 \pmod{10000}.$$

Hence, $-N = 2007 - 10000 = -7993 \quad \Rightarrow \quad N = 7993$.

5. How many positive 5-digit odd integers are palindromes?

Answer: 500

Solution: A 5-digit palindrome is in the form $abcba$ for digits a, b, and c. In this case, a must be odd. So, there are 5 choices for a, 10 choices for b, and 10 choices for c. The total number of odd 5-digit palindromes is $5 \cdot 10 \cdot 10 = 500$.

6. If r is a root of $x^2 + x + 6$, then compute the value of
$$r^3 + 2r^2 + 7r + 17.$$

Answer: 11

Solution: We want to relate the given cubic to the equation $r^2 + r + 6$. More specifically, we'd like to construct the cubic using the given quadratic. We apply the Division Theorem for Polynomials (we divide the cubic by the quadratic):
$$r^3 + 2r^2 + 7r + 17 = (r + 1)(r^2 + r + 6) + 11,$$
where $r + 1$ is the quotient, and 11 is the remainder. Now we see clearly that
$$r^3 + 2r^2 + 7r + 17 = (r + 1)(0) + 11 = 11.$$

7. Find the number of <u>distinct</u> integers in the list
$$\left\lfloor \frac{1^2}{2007} \right\rfloor, \left\lfloor \frac{2^2}{2007} \right\rfloor, \left\lfloor \frac{3^2}{2007} \right\rfloor, \left\lfloor \frac{4^2}{2007} \right\rfloor, \ldots, \left\lfloor \frac{2007^2}{2007} \right\rfloor,$$
where $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

Answer: 1506

Solution: The first several terms equal 0 – the 45^{th} term is the first to be greater than 0:
$$\left\lfloor \frac{45^2}{2007} \right\rfloor = \left\lfloor \frac{2025}{2007} \right\rfloor = \left\lfloor 1 + \frac{18}{2007} \right\rfloor = 1.$$
It is often useful to identify the important function in a problem. In this case we notice that where
$$g(n) = \left\lfloor \frac{n^2}{2007} \right\rfloor,$$
the terms in our list are $g(1), g(2), g(3), \ldots, g(2007)$. For a while, this function increases slowly, and overall, it increases *about* as fast as $f(x) = x^2/2007$. In fact, for $x \leq 1004$,
$$f(x) - f(x - 1) \leq 1,$$
and thus
$$g(x) - g(x - 1) \leq 1.$$
In other words, the difference between consecutive terms of each sequence is no greater than 1. (Note: $1004^2 - 1003^2 = (1004 + 1003)(1004 - 1003) = 2007 \cdot 1 = 2007$, which is why we only looking at $x \leq 1004$ so far.)

Now, we note that
$$\begin{aligned} g(1004) &= \left\lfloor \frac{1004^2}{2007} \right\rfloor = \left\lfloor \frac{502 \cdot 2008}{2007} \right\rfloor \\ &= \left\lfloor 502 \left(\frac{2008}{2007} \right) \right\rfloor = \left\lfloor 502 \left(1 + \frac{1}{2007} \right) \right\rfloor \\ &= \left\lfloor 502 + \frac{502}{2007} \right\rfloor = 502. \end{aligned}$$

So, the first 1004 terms in the given list are

$$0, 0, 0, 0, \ldots, 0, 1, 1, 1, \ldots, 502,$$

where the difference between consecutive terms is no greater than 1.

From here on out, the nature of the list changes,

$$f(x) - f(x-1) > 1,$$

for $x > 1004$. Thus,

$$g(x) - g(x-1) \geq 1.$$

This means that after the 1004$^{\text{th}}$ term in the list, the rest of the terms are distinct. In other words, the last 1003 terms are distinct.

So, our list consists of the 503 distinct integers from 0 to 502 inclusive, plus 1003 more distinct integers, for a total of $503 + 1003 = 1506$ distinct integers.

8. Krugman is a little odd. Really odd actually. He has a collection of Massachusetts state quarters and New Hampshire state quarters. All 29 of his Massachusetts quarters are kept face up, while all 11 of his New Hampshire quarters are kept face down on display in his office. Eric, being the prankster that he is, sneaks into Krugman's office and randomly turns over 20 of Krugman's quarters. Find the expected number of Krugman's quarters that are heads up when Eric is finished.

   Answer: 20

   Solution: There are $11+29 = 40$ total quarters. Eric flips over exactly half of them. Another way of looking at it is that each quarter has a 1/2 probability of being turned over. Each of the 40 quarters is just as likely to be heads as tails when Eric is done. So, the expected number of quarters showing heads is $40/2 = 20$.

9. Let $F_1 = F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. Find the value of k such that $x = F_k$ is the x-coordinate of the intersection between the linear equations

$$F_{2007}x + F_{2008}y = F_4,$$
$$F_{2008}x + F_{2009}y = -F_3.$$

   Answer: 2012

   Solution: We want to rid ourselves of the y variable in the given system of equations. We do so by elimination. We multiply the first equation by F_{2009}, the second equation by F_{2008}, and subtract the results to get

$$(F_{2007}F_{2009} - F_{2008}^2)x = F_4F_{2009} + F_3F_{2008}.$$

   Now, we simplify the right-hand side of our new single-variable equation by noting that $F_4 = 3$, $F_3 = 2$, and applying the recursive definition of the Fibonacci sequence:

$$\begin{aligned} F_4F_{2009} + F_3F_{2008} &= 3F_{2009} + 2F_{2008}, \\ &= F_{2009} + 2(F_{2008} + F_{2009}) = F_{2009} + 2F_{2010}, \\ &= (F_{2009} + F_{2010}) + F_{2010} = F_{2011} + F_{2010} = F_{2012}. \end{aligned}$$

So, our equation becomes

$$(F_{2007}F_{2009} - F_{2008}^2)x = F_{2012}.$$

Simplifying the left-hand side of this equation is more difficult. In order to get a grip on its possible value, we examine similar quantities in which the Fibonacci numbers are smaller:

$$\begin{aligned}
F_1F_3 - F_2^2 &= \quad 1 \cdot 2 - 1^2 = \quad 1, \\
F_2F_4 - F_3^2 &= \quad 1 \cdot 3 - 2^2 = -1, \\
F_3F_5 - F_4^2 &= \quad 2 \cdot 5 - 3^2 = \quad 1, \\
F_4F_6 - F_5^2 &= \ 3 \cdot 8 - 25^2 = -1, \\
F_5F_7 - F_6^2 &= \ 5 \cdot 13 - 8^2 = \quad 1, \\
F_6F_8 - F_7^2 &= 8 \cdot 21 - 13^2 = -1.
\end{aligned}$$

We could be satisfied with the pattern we see and produce the correct answer:

$$(F_{2007}F_{2009} - F_{2008}^2)x = x = F_{2012},$$

so $k = 2012$. We could also prove the observed pattern using induction. There are several ways to establish the inductive step, but the easiest might be noting that

$$\begin{aligned}
F_kF_{k+2} - F_{k+1}^2 + F_{k+1}F_{k+3} - F_{k+2}^2 &= F_{k+2}(F_k - F_{k+2}) + F_{k+1}(F_{k+3} - F_{k+1}) \\
&= -F_{k+2}F_{k+1} + F_{k+1}F_{k+2} = 0,
\end{aligned}$$

which proves the alternation of the signs of the series of quantities above.

10. Andrew and Patri each have a deck of cards. Andrew draws two cards from his deck, both of which turn out to be queens. Patri draws two cards from his deck, one of which is an ace, and the other of which is a king. Andrew and Patri then put the remainder of their decks (50 cards left in each deck) down on a table and race to the nearest river. Meanwhile, Chelsea comes along and picks up one of the decks at random. She randomly pulls two more cards out of that deck. It turns out Chelsea's cards make a pair (both of the same rank, i.e., both are 2's, both are 3's, etc.). Find the probability that she drew those two cards from Andrew's deck.

Answer: $\dfrac{73}{145}$

Solution: A dependent probability problem such as this one becomes much more clear if we first find the probabilities associated with each *state*. Let p be the probability that Chelsea draws a pair from Andrew's deck (when drawing 2 cards), and q be the probability that Chelsea draws a pair from Patri's deck (when drawing 2 cards).

We compute p and q using an ordinary counting probability strategy, with a tiny bit of casework. First, let's note the cases:

(i) If there are 4 cards in a rank remaining, then there are $\binom{4}{2} = 6$ ways to draw a pair of that rank.

(ii) If there are 3 cards in a rank remaining, then there are $\binom{3}{2} = 3$ ways to draw a pair of that rank.

(iii) If there are 2 cards in a rank remaining, then there is $\binom{2}{2} = 1$ way to draw a pair of that rank.

Also, there are 50 cards left in each deck, so the number of pairs of cards that could be drawn from either is $\binom{50}{2} = 50 \cdot 49/2 = 1225$. Though, as we shall see, we need not even compute this number.

To compute p, we note that in Andrew's deck, there are 12 ranks (all but the queen) with all 4 cards in that rank, and one rank (queens) with just 2 cards remaining. So, we have 12 ranks of case (i) and 1 rank of case (iii) for a total of

$$12 \cdot 6 + 1 \cdot 1 = 73$$

possible pairs. Hence, $p = 73/1225$.

To compute q, we note that in Patri's deck, there are 11 ranks (not aces or kings) with all 4 cards remaining in that rank, and two ranks (aces and kings) with 3 cards remaining. So, we have 11 ranks of case (i) and 2 ranks of case (ii) for a total of

$$11 \cdot 6 + 2 \cdot 3 = 72$$

possible pairs. Hence, $q = 72/1225$.

Finally, we apply dependent probability to find the probability that Chelsea's pair came from Andrew's deck:

$$\frac{p}{p+q} = \frac{\frac{73}{1225}}{\frac{73}{1225} + \frac{72}{1225}} = \frac{73}{73+72} = \frac{73}{145}.$$

Note that the last bit of simplification of the above fractions makes it clear that we didn't need to compute $\binom{50}{2} = 1225$. All that mattered were the ratios. A student with experience solving this kind of problem might have jumped straight into computation as follows:

$$\frac{12 \cdot \binom{4}{2} + 1 \cdot \binom{2}{2}}{(12+11) \cdot \binom{4}{2} + 2 \cdot \binom{3}{2} + 1 \cdot \binom{2}{2}}.$$

11. In how many distinct ways can a rectangular 3×17 grid be tiled with 17 non-overlapping 1×3 rectangular tiles?

Answer: 406

Solution: Often, the best way to tackle a problem is to tackle simpler cases. In this case, we begin by examining the numbers of ways to tile $3 \times n$ grids for small values of n. For visual purposes, we consider a $3 \times n$ grid to be a grid 3 units high (vertically) and n units across (horizontally).

Let a_n be the number of ways we can tile a $3 \times n$ grid with 1×3 tiles. When $n = 1$ or $n = 2$ there is clearly only one way to tile each grid: with 1 or 2 vertically placed 1×3 tiles, respectively. So, $a_1 = a_2 = 1$. However, when $n = 3$, there are two ways: with three verticle tiles or three horizontal tiles. So, $a_3 = 2$.

Now, in constructing larger $3 \times n$ grids, we note that there are two ways to *end* each grid (to place tiles on the far right side). We can either take any $3 \times (n-3)$ grid and add three horizontally stacked 1×3 tiles, or we can take any $3 \times (n-1)$ grid and add one vertical 1×3 tile. This "either-or" construction of $3 \times n$ grid tilings allows us to express a_n recursively:

$$a_n = a_{n-1} + a_{n-3}, \quad n \leq 4.$$

Finally, we compute:

$$\begin{array}{rcccccl}
a_4 &=& a_3 + a_1 &=& 2+1 &=& 3, \\
a_5 &=& a_4 + a_2 &=& 3+1 &=& 4, \\
a_6 &=& a_5 + a_3 &=& 4+2 &=& 6, \\
a_7 &=& a_6 + a_4 &=& 6+3 &=& 9, \\
a_8 &=& a_7 + a_5 &=& 9+4 &=& 13, \\
a_9 &=& a_8 + a_6 &=& 13+6 &=& 19, \\
a_{10} &=& a_9 + a_7 &=& 19+9 &=& 28, \\
a_{11} &=& a_{10} + a_8 &=& 28+13 &=& 41, \\
a_{12} &=& a_{11} + a_9 &=& 41+19 &=& 60, \\
a_{13} &=& a_{12} + a_{10} &=& 60+28 &=& 88, \\
a_{14} &=& a_{13} + a_{11} &=& 88+41 &=& 129, \\
a_{15} &=& a_{14} + a_{12} &=& 129+60 &=& 189, \\
a_{16} &=& a_{15} + a_{13} &=& 189+88 &=& 277, \\
a_{17} &=& a_{16} + a_{14} &=& 277+129 &=& 406.
\end{array}$$

12. If $\omega^{2007} = 1$ and $\omega \neq 1$, then evaluate

$$\frac{1}{1+\omega} + \frac{1}{1+\omega^2} + \frac{1}{1+\omega^3} + \cdots + \frac{1}{1+\omega^{2007}}.$$

Express your answer as a fraction in lowest terms.

Answer: $\dfrac{2007}{2}$

Solution: The last term is itself easy to evaluate:

$$\frac{1}{1+\omega^{2007}} = \frac{1}{1+1} = \frac{1}{2}.$$

The other terms prove a little trickier. Experienced series smashers might try pairing up remaining terms (from the ends) and find the following:

$$\begin{aligned}
\frac{1}{1+\omega} + \frac{1}{1+\omega^{2006}} &= \frac{\left(1+\omega^{2006}\right) + (1+\omega)}{(1+\omega)\left(1+\omega^{2006}\right)} \\
&= \frac{2+\omega+\omega^{2006}}{1+\omega+\omega^{2006}+\omega^{2007}} \\
&= \frac{2+\omega+\omega^{2006}}{2+\omega+\omega^{2006}} = 1.
\end{aligned}$$

More generally,

$$\frac{1}{1+\omega^n}+\frac{1}{1+\omega^{2007-n}}=1$$

for $1 \le n \le 1003$. So, the first 2006 terms of the series pair up to 1003 pairs that each sum to 1. The total value of the series is thus $1003+\frac{1}{2}=\frac{2007}{2}$.

13. Before he gets out of bed every morning, Calvin the Compulsive plays a game with a fair coin. He flips it until *either* he flips four consecutive heads *or* he flips six consecutive tails, then he immediately gets out of bed and brushes his teeth. If his last flip is a head, he eats two melons for breakfast. Otherwise, he eats just one. Find the probability that Calvin ate two melons for breakfast this morning.

    Answer: $\dfrac{21}{26}$

    Solution: We define two "states" for the game Calvin plays. Once Calvin has flipped the coin once, he is either in a *head state* or a *tail state*, according to the result of his last flip. So, we note that once Calvin has entered a head state (flipped one head), he has a $(1/2)^3 = 1/8$ chance of ending the game with four consecutive heads. Similarly, once Calvin has entered a tail state (flipped a tail), he has a $(1/2)^5 = 1/32$ chance of ending the game with six consecutive tails.

    When Cal flips a head on his first flip, his chance of ending the game with a head is

$$S=\frac{1}{8}+\left(\frac{7}{8}\cdot\frac{31}{32}\right)\cdot\frac{7}{8}+\left(\frac{7}{8}\cdot\frac{31}{32}\right)^2\cdot\frac{7}{8}+\left(\frac{7}{8}\cdot\frac{31}{32}\right)^3\cdot\frac{7}{8}+\left(\frac{7}{8}\cdot\frac{31}{32}\right)^4\cdot\frac{7}{8}+\cdots.$$

    When he clips a tail first, his chance of ending the game with a head is

$$\left(1-\frac{1}{32}\right)S=\frac{31}{32}S.$$

    Using the formula for the value of an infinite geometric series, we find that the chance of Cal eating two melons (ending the game with a head) is

$$\begin{aligned}\frac{1}{2}S+\frac{1}{2}\cdot\frac{31}{32}S &= \frac{63}{64}S=\frac{63}{64}\cdot\frac{\frac{1}{8}}{1-\frac{7}{8}\cdot\frac{31}{32}} \\ &= \frac{63}{64}\cdot\frac{\frac{1}{8}}{1-\frac{217}{256}}=\frac{63}{64}\cdot\frac{\frac{1}{8}}{\frac{39}{256}} \\ &= \frac{63}{64}\cdot\frac{32}{39}=\frac{21}{26}.\end{aligned}$$

14. Find the sum of the real roots of

$$x^6+145x^4-2007x^3+145x^2+1=0.$$

    Answer: 9

Solution: The interesting part of the sixth degree polynomial is the symmetry of the coefficients. Dividing the given equation by x^3 allows us to take advantage of that symmetry while preserving the roots, none of which are $x = 0$:

$$x^3 + 145x - 2007 + 145x^{-1} + x^{-3} = 0.$$

Now, we consider the substitution $y = x + \frac{1}{x}$, which will help us reduce the degree of the given polynomial to 3. First, we note that

$$y^3 = x^3 + 3x + 3x^{-1} + x^{-3},$$

so

$$x^3 + 145x - 2007 + 145x^{-1} + x^{-3} = y^3 + 142y - 2007 = 0.$$

Now, we might as well look for rational roots to $y^3 + 142y - 2007$. Doing so, we find that $y = 9$ is a root. Now, we factor to see that

$$y^3 + 142y - 2007 = (y - 9)(y^2 + 9y + 223) = 0.$$

The roots of $y^2 + 9y + 223$ are (nonreal) complex since the discriminant $9^2 - 4(223) < 0$. When x is real, $y = x + \frac{1}{x}$ is necessarily real, so $y = 9$ is the only root we are interested in. So,

$$y = x + \frac{1}{x} = 9 \qquad \Rightarrow \qquad x^2 - 9x + 1 = 0.$$

The discriminant of this new quadratic is $9^2 - 4(1) > 0$, so both roots are real. By Vieta, their sum is $-(-9)/1 = 9$.

15. Let P be a point inside isosceles right triangle ABC such that $\angle C = 90°$, $AP = 5$, $BP = 13$, and $CP = 6\sqrt{2}$. Find the area of ABC.

Answer: $\dfrac{157}{2}$

Solution: With many harder geometry problems, it helps to think outside the initial diagram. In this case, we start with triangle ABC with A on the y-axis, B on the x-axis, and C at the origin (just to give us a starting position for the diagram – coordinates are not really part of the solution). Now, rotating the triangle counterclockwise $90°$ about C brings point B to A, and A to a new point A'. We also label P', where P lands after our rotation.

Note that $\angle PCP' = 90°$ and $PC = P'C$, so $\triangle PCP'$ is right and isosceles. Since $PC = 6\sqrt{2}$, we see that $PP' = 12$. Labeling all the lengths we know in the diagram, we now see that APP' is 5-12-13 triangle, so $\angle APP' = 90°$. This breaks the problem open as we now note that

$$\angle APC = \angle APP' + \angle P'PC = 90° + 45° = 135°.$$

Now we can apply the Law of Cosines to find AC:

$$AC^2 = 5^2 + (6\sqrt{2})^2 - 2(5)(6\sqrt{2})\cos 135° = 25 + 72 - 60\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = 25 + 72 + 60 = 157.$$

The area of ABC is half the square of the leg length, which is $157/2$.