MIST
Academy

Number Theory

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Number Theory is the branch of mathematics associated with the integers. This packet includes a wide variety of number theory problems as well as a number of helpful solutions. The concepts covered test a wide variety of students. Do not be discouraged if you cannot yet complete any given section. The sections are not exactly ordered by difficulty either, and problems at the end of one section might be considerably harder than the earlier problems in the next.

Try to solve as many problems as you can in each section. Along the way you might learn new concepts that help you with some of the problems you skipped.

Unfortunately, this packet cannot replace a good Number Theory textbook, but few such books have been written for excellent high school students. It should however help you learn, relearn, or dig deeper into a number of important mathematical concepts.

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Counting Numbers

To begin with, we use integers for counting. In particular, we use the *counting numbers* or *natural numbers* for counting:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, \ldots$$

1. How many of the first one hundred positive integers are divisible by all of the numbers 2, 3, 4, 5?

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

   AHSME

2. In how many ways can 16 rocks be divided into 5 piles such that the total number of rocks in each pile is odd?

3. How many multiples of 72 are there between 1000 and 10000 inclusive?

   Solution: We can manipulate the integers from 1000 to 10000 as a list:

   $$1000, 1001, 1002, 1003, \ldots, 9999, 10000.$$

   Dividing each integer in this list by 72, we get

   $$13\frac{64}{72}, 13\frac{65}{72}, 13\frac{66}{72}, 13\frac{67}{72}, \ldots, 138\frac{63}{72}, 138\frac{64}{72}.$$

   Now, we remove the non-integers from our list, leaving us with one term for each multiple of 72 in the original list:

   $$14, 15, 16, 17, \ldots, 137, 138.$$

   Finally, we subtract 13 from each integer in order list, giving us the simplest possible list to count without changing the number of terms (which is still equal to the number of multiples of 72 from 1000 to 10000 inclusive):

   $$1, 2, 3, 4, \ldots, 124, 125.$$

   There are 125 integers in this list representing the 125 multiples of 72 from 1000 to 10000 inclusive.

4. How many perfect squares are there between 300 and 500?

5. For how many positive integers n is $9 < \sqrt{n} < 10$? *Mandelbrot*

6. Jerry counted to 600 by 6's, beginning with 6, while Jenny counted to 600 by 4's, beginning with 4. How many of the numbers counted by Jerry were also counted by Jenny?

7. The diameter of a circle is a whole number. The area of the circle is between 100 and 120 square units. What is the number of units in the circle's diameter? *MATHCOUNTS*

8. How many integers between 2 and 120 are perfect powers?

9. Find all the perfect cubes between 2000 and 3000.

Prime Numbers

1. Practice the Sieve of Eratosthenes to find all the prime between 1 and 200.

2. Three primes, p, q and r, satisfy $p + q = r$ and $1 < p < q$. Then p equals *AHSME*

   (A) 2 (B) 3 (C) 7 (D) 13 (E) 17

3. A **twin prime pair** is a pair of primes (p, q) such that $q = p + 2$. The **Twin Prime Conjecture** states that there are infinitely many twin prime pairs. What is the arithmetic mean of the two primes in the smallest twin prime pair? (1 is not a prime.) *iTest*

   Answer: 4

   Solution: The smallest twin prime pair is (3, 5). The arithmetic mean of 3 and 5 is 4.

4. How many prime numbers are multiples of 7?

5. How many prime numbers are multiples of 20?

   Solution: A multiple of 20 must be of the form $20n$ for some integer n. Since the divisors of $20n$ include all the divisors of 20 (1, 2, 4, 5, 10, 20), $20n$ is not prime. That is, there are no prime multiples of 20.

6. What is the largest two-digit prime number whose digits are also prime? *MATHCOUNTS*

7. Is the sum of the 30 smallest primes even or odd?

8. Paul would like to find out if 1147 is prime or composite, but he's not very interested in testing all the primes less than 1147 to see which might be divisors of 1147. Help Paul by finding a quicker method that determines whether or not an integer is prime.

9. Which of the following integers are prime and which are composite?

   (a) 217
   (b) 233
   (c) 343
   (d) 451
   (e) 1003
   (f) 1927

10. Find the 4 smallest primes that are all greater than 1000.

    Solution: There is no need to perform the entire Sieve of Eratosthenes. We can begin by listing multiples of the several smallest primes:

Multiples of 2:	1002, 1004, 1006, 1008, 1010, 1012, 1014, ...
Multiples of 3:	1002, 1005, 1008, 1011, 1014, 1017, 1020, ...
Multiples of 5:	1005, 1010, 1015, 1020, 1025, 1030, 1035, ...
Multiples of 7:	1001, 1008, 1015, 1022, 1029, 1036, 1043, ...
Multiples of 11:	1001, 1012, 1023, 1034, 1045, 1056, 1067, ...

    The smallest integers greater than 1000 that aren't in any of these lists are 1003, 1007, 1009, and 1013. Now we can test each of these for prime divisors from 13 to $\lfloor\sqrt{1013}\rfloor = 31$. We find that $1003 = 17 \cdot 59$ and $1007 = 19 \cdot 53$, but 1009 and 1013 are both prime. The next primes to try are 1019 and 1021, which turn out to be prime.

It might be even faster to take note of a multiple of each of the larger primes less than or equal to 31:

$$\begin{aligned}1001 &= 7 \cdot 11 \cdot 13,\\ 1020 &= 17 \cdot 60,\\ 1007 &= 19 \cdot 53,\\ 1012 &= 23 \cdot 44,\\ 1015 &= 29 \cdot 35,\\ 1023 &= 31 \cdot 33.\end{aligned}$$

Make sure you see why.

11. Find the prime factorization of 999,999.

12. Find the greatest prime divisor of the arithmetic series

$$1 + 2 + 3 + \cdots + 91.$$

Solution: Using the formula for the sum of the first n positive integers, we note that

$$1 + 2 + 3 + \cdots + 91 = \frac{91 \cdot 92}{2}.$$

Now, we could compute this sum, but manipulating the product on the right-hand side into a prime factorization gives us exactly what we want:

$$\frac{91 \cdot 92}{2} = \frac{7 \cdot 13 \cdot 2^2 \cdot 23}{2} = 2^1 \cdot 7^1 \cdot 13^1 \cdot 23^1,$$

so 23 is the greatest prime divisor of the given sum.

13. Express each of the following as a single fraction in lowest terms:

(a) $\frac{540}{162}$ (b) $\frac{18}{1296}$

(c) $-\frac{2520}{3360}$ (d) $\frac{2^3 \cdot 3^6 \cdot 5^2 \cdot 7^1}{2^5 \cdot 3^4 \cdot 5^1 \cdot 7^2}$

14. Determine each of the following using prime factorization.

(a) $\gcd(18, 24)$ (b) $\text{lcm}[18, 24]$
(c) $\gcd(48, 84)$ (d) $\text{lcm}[48, 84]$
(e) $\gcd(200, 300)$ (f) $\text{lcm}[200, 300]$
(g) $\gcd(630, 147)$ (h) $\text{lcm}[630, 147]$

15. Let m and n be positive integers. Find a general relationship between m, n, their GCD, and their LCM.

16. Find the product of two natural numbers whose GCD is 18 and whose LCM is 3240.

17. Find the five smallest positive multiples of 15, 28, and 80.

    Solution: We note the prime factorizations of the given integers:

    $$\begin{aligned} 15 &= 3^1 \cdot 5^1, \\ 28 &= 2^2 \cdot 7^1, \\ 80 &= 2^4 \cdot 5^1. \end{aligned}$$

    A multiple of each of these integers must be a multiple of the largest powers of each of the primes in these factorizations, thus the common multiples are of the form

    $$2^4 \cdot 3^1 \cdot 5^1 \cdot 7^1 \cdot n = 1680n,$$

    for each integer n. The five smallest positive multiples are thus

    $$\begin{aligned} 1680 \cdot 1 &= 1680, \\ 1680 \cdot 2 &= 3360, \\ 1680 \cdot 3 &= 5040, \\ 1680 \cdot 4 &= 6720, \\ 1680 \cdot 5 &= 8400. \end{aligned}$$

18. Jerry asks Elaine to find a positive integer n less than 25 satisfying the conditions that n is even, n is prime, and the sum of the digits of n is 7. The best Elaine can do is to find positive integers j, k, and l less than 25 such that j satisfies the first two conditions, k satisfies the first and third conditions, and l satisfies the second and third. Find the sum $j + k + l$. *Mandelbrot*

19. Walter rolls four standard six-sided dice and finds that the product of the numbers on the upper faces is 144. Which of the following could **not** be the sum of the upper four faces?

    (A) 14 (B) 15 (C) 16 (D) 17 (E) 18

    AHSME

20. The LCM of 15, 18, 35, and k is 2520. Find the smallest possible value of k.

The Euclidean Algorithm

The Euclidean Algorithm is a computationally efficient (quick) way to find the greatest common divisor (GCD) of a pair of integers. Let's take a look at an example, and then rehash the process:

$$\gcd(18, 24) = \gcd(18, 24-18) = \gcd(18, 6) = \gcd(18-6, 6) = \gcd(12, 6) = \gcd(12-6, 6) = \gcd(6, 6).$$

So, the GCD of 18 and 24 is 6. We started with the integers 18 and 24. We replaced (the larger number) 24 with $24 - 18$ to get 6. Then we replaced the larger of 18 and 6 with $18 - 6$. We repeated this step once more and came up with a pair of 6's.

See if you can prove why this works on your own. It is important to understand the process, how to shorten it, and when you have mastered it, how it applied to polynomials as well!

Number Theory

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1. Determine each of the following using the Euclidean Algorithm.

(a)	$\gcd(126, 24)$	(c)	$\text{lcm}[572, 1331]$
(b)	$\gcd(1458, 441)$	(d)	$\text{lcm}[96, 484848]$

2. In the game of *Winners Make Zeros*, a pair of positive integers (m, n) is written on a sheet of paper. Then the game begins, as the players make the following legal moves:

   - If $m \geq n$, the player chooses a positive integer c such that $m - cn \geq 0$, and replaces (m, n) with $(m - cn, n)$.
   - If $m < n$, the player chooses a positive integer c such that $n - cm \geq 0$, and replaces (m, n) with $(m, n - cm)$.

   When m or n becomes 0, the game ends, and the last player to have moved is declared the winner. If m and n are originally 2007777 and 2007, find the largest choice the first player can make for c (on his first move) such that the first player has a winning strategy after that first move. *iTest TOC*

   Answer: 999

   Credit: This problem is based on the game usually known as *Euclid*.

   Solution: First, we note that the game jumps through "steps" in the Euclidean Algorithm.

   Now, note that from any position, a player cannot have multiple moves that lead to winning positions. If $c = c_1$ and $c = c_2$ lead to winning positions, where $c_1 > c_2$, then after choosing c_2, a players opponent could let $c = c_1 - c_2$ to take a winning position, contradicting the fact that both $c = c_1$ and $c = c_2$ both lead to winning positions. So, we are looking for the one value of c that the first player can use to leave himself with a winning strategy.

   Next, since the game is necessarily finite, we can construct all possible sets of moves. This means that a player either has a winning strategy, or they are in a losing position (so long as their opponent plays perfectly). However, if a player has more than one move, they cannot be in a losing position, because they could always choose $c = 1$, leaving their opponent with a subset of their available moves. This means that a player with more than one legal move necessarily has a winning position.

   Now, let's take a look at parts of the Euclidean Algorithm that take us from the beginning of the game to the end, noting each point at which a player's move would be forced (because

$c = 1$ leads to the only legal move):

(m, n)	Next Move Forced
$(2007777, 2007)$	
$(2005770, 2007)$	
$\vdots$	
$(4791, 2007)$	
$(2784, 2007)$	✓
$(777, 2007)$	
$(777, 1230)$	✓
$(777, 453)$	✓
$(324, 453)$	✓
$(324, 129)$	
$(195, 129)$	✓
$(66, 129)$	✓
$(66, 63)$	✓
$(3, 63)$	

The player who gets to move when $(m, n) = (3, 63)$ wins by replacing $(3, 63)$ with $(3, 0)$. Walking backward through the moves, the player with the winning strategy wants to avoid having their turn when $(m, n) = (66, 63)$. This means they want to have their turn when $(m, n) = (66, 129)$. They want to avoid $(m, n) = (195, 129)$. They want their turn when $(m, n) = (324, 129)$. They want to avoid $(m, n) = (324, 453)$. They want their turn when $(m, n) = (777, 453)$. They want to avoid $(m, n) = (777, 1230)$. They want their turn when $(m, n) = (777, 2007)$. So, $(m, n) = (2784, 2007)$ is a losing position, where

$$2784 = 2007777 - c \cdot 2007 \qquad \Rightarrow \qquad c = 999.$$

Factorials

1. $\dfrac{(3!)!}{3!} =$

   (A) 1 (B) 2 (C) 6 (D) 40 (E) 120

   AHSME

2. Find the largest prime divisor of $18! + 19!$.

3. In how many zeros does $143!$ end?

   Solution: Understanding the decimal system leads us to a nice reinterpretation of this problem. The number of zeros in which $143!$ ends is exactly the number of powers of 10 that divide $143!$. Since $10 = 2 \cdot 5$, we are really asking what is the minimum value of a and b, where $2^a \cdot 5^b$ is included in the prime factorization of $143!$.

   Note: **Legendre's Theorem** tells us that the power to a prime p in the prime factorization of $n!$ is

$$\left\lfloor \frac{n}{p^1} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots .$$

If you are familiar with this theorem, it is important that you learn how to prove it. A complete understanding of this theorem helps to break down *many* difficult number theory problems.

Now, when $n = 143$ (or any other natural number), it should be clear that when $p = 2$, the "Legendre Series" above will be no larger than when $p = 5$, so we just evaluate the series for $p = 5$:

$$\begin{aligned}
\left\lfloor \frac{143}{5^1} \right\rfloor + \left\lfloor \frac{143}{5^2} \right\rfloor + \left\lfloor \frac{143}{5^3} \right\rfloor + \cdots &= \left\lfloor 28.6 \right\rfloor + \left\lfloor \frac{28.6}{5^1} \right\rfloor + \left\lfloor \frac{28.6}{5^2} \right\rfloor + 0 + 0 + \cdots \\
&= 28 + \left\lfloor \frac{28}{5^1} \right\rfloor + \left\lfloor \frac{28}{5^2} \right\rfloor + 0 + 0 + \cdots \\
&= 28 + \left\lfloor 5.6 \right\rfloor + \left\lfloor \frac{5.6}{5} \right\rfloor + 0 + 0 + \cdots \\
&= 28 + 5 + \left\lfloor \frac{5}{5} \right\rfloor + 0 + 0 + \cdots \\
&= 28 + 5 + 1 + 0 + 0 + \cdots = 34.
\end{aligned}$$

Make sure you understand the steps taken above to quicken computation.

So, there are 34 powers of 5 that divide 143!, and at least that many powers of 2. Thus, there are 34 powers of 10 that divide 143!, which is therefore an integer with 34 terminal zeros.

4. In how many terminal zeros does $311 \cdot 312 \cdot 313 \cdot 314 \cdots 417 \cdot 418$ end?

5. Suppose a, b, and c are positive integers with $a + b + c = 2006$, and $a!b!c! = m \cdot 10^n$, where m and n are integers and m is not divisible by 10. What is the smallest possible value of n?

   (A) 489 (B) 492 (C) 495 (D) 498 (E) 501

6. Find the largest integer n for which 12^n evenly divides 100!.

   Note: What makes this problem different from the problem of finding the number of terminal zeros is that *twice as many* powers of 2 are needed as powers of 3.

7. If, from left to right, the last seven digits of $n!$ are 8000000, compute the value of n. *ARML*

8. There are unique integers a_2, a_3, a_4, a_5, a_6, a_7 such that

$$\frac{5}{7} = \frac{a_2}{2!} + \frac{a_3}{3!} + \frac{a_4}{4!} + \frac{a_5}{5!} + \frac{a_6}{6!} + \frac{a_7}{7!},$$

   where $0 \le a_i < i$ for $i = 2, 3, \ldots, 7$. Find $a_2 + a_3 + a_4 + a_5 + a_6 + a_7$.

   (A) 8 (B) 9 (C) 10 (D) 11 (E) 12

   AHSME

9. For each positive integer n, let

$$a_n = \frac{(n+9)!}{(n-1)!}$$

   Let k denote the smallest positive integer for which the rightmost nonzero digit of a_k is odd. The rightmost nonzero digit of a_k is

   (A) 1 (B) 3 (C) 5 (D) 7 (E) 9

   AHSME

Perfect, Abundant, and Deficient Numbers

Let $s(n)$ be the sum of the divisors of a natural number n. Then,

- A **perfect number** is a natural number, the sum of whose divisors is twice the number itself. For instance, 6 is a perfect number:

$$s(6) = 1 + 2 + 3 + 6 = 12 = 2 \cdot 6.$$

- An **abundant number** is a natural number, the sum of whose divisors is greater than twice the number itself. For instance, 12 is an abundant number:

$$s(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28 > 2 \cdot 12.$$

- A **deficient number** is a natural number, the sum of whose divisors is less than twice the number itself. For instance, 8 is a deficient number:

$$s(8) = 1 + 2 + 4 + 8 = 15 < 2 \cdot 8.$$

1. Which one of the following natural numbers is an abundant number: 14, 28, or 56? *iTest*

   Answer: 56

   Solution 1: We find and add together the proper divisors of each integer:

$$\begin{aligned} 1 + 2 + 7 &= 10 < 14, \\ 1 + 2 + 4 + 7 + 14 &= 28 = 28, \\ 1 + 2 + 4 + 7 + 8 + 14 + 28 &= 64 > 56. \end{aligned}$$

   As we see, the sum of the proper divisors of 14 is less than 14, so 14 is not abundant. The sum of the proper divisors of 28 is equal to 28. The sum of the proper divisors of 56 is greater than 56, so 56 is an abundant number.

   Solution 2: Some students might notice a relationship between the divisors of the given related integers:

$$1 + 2 + 4 + 7 + 14 = (1 + 7) + (2 + 4 + 14) = (1 + 7) + 2(1 + 2 + 7) = 8 + 2(10) = 28.$$

   Here, we view the proper divisors of 28 in terms of all the proper divisors of 14. Since $28 = 2 \cdot 14$, each proper divisor of 14 corresponds to a proper divisor of 28 which is twice as large. Adding in the odd proper divisors of 28, we see that the sum of the proper divisors of 28 is more than twice the sum of the proper divisors of 14. Likewise, the sum of the proper divisors of 56 is more than twice the sum of the proper divisors of 28. So, if only one of the three integers is abundant, it must be 56.

   How far can you generalize this relationship?

2. Label each of the following integers as perfect, abundant, or deficient:

   (a) 12
   (b) 120
   (c) 18
   (d) 7
   (e) 27
   (f) 182

3. a. Show that all prime numbers are deficient.
   b. Show that for all primes p and natural numbers n, that p^n is deficient.
4. Prove that any positive multiple of an abundant number is also abundant.
5. Prove that any divisor of a deficient number is also deficient.

Palindromes

A **palindrome** is an integer that reads the same forward as it does when its digits are read backward. Some palindromes have interesting properties that make them good candidates for interesting problems.

1. a. How many two-digit palindromes are prime?
   b. How many three-digit palindromes are prime?
2. The product of two positive three-digit palindromes is 436995. What is their sum? *MATHCOUNTS*
3. Find each of the following:
   (a) The number of 3-digit palindromes.
   (b) The number of 4-digit palindromes.
   (c) The number of 5-digit palindromes.
4. Before Ashley started a two-hour drive, her car's odometer reading was 27972, a palindrome. (A palindrome is a number that reads the same way from left to right as it does from right to left.) At her destination, the odometer reading was another palindrome. If Ashley never exceeded the speed limit of 75 miles per hour, which of the following was her average speed?

   (A) 50 (B) 55 (C) 60 (D) 65 (E) 70

   AHSME
5. Find the smallest possible GCD of
   a. a pair of five-digit palindromes.
   b. a pair of six-digit palindromes.
6. The sum of two 4-digit palindromes is the 5-digit palindrome N. Compute the maximum possible value of N. *ARML*

Special Primes

- A **Mersenne prime** is a prime number of the form $2^p - 1$, where p is prime.

- A **Fermat prime** is a prime number of the form $2^{2^n} + 1$ for some whole number n.
- A **twin prime** pair is a pair of prime numbers that differ by 2.

1. Identify some examples of each of the special primes listed above.
2. Identify which of the following are prime:
   (a) $2^7 - 1$
   (b) $2^{11} - 1$
   (c) $2^{2^3} + 1$
   (d) $2^{2^5} + 1$ (hard)
3. Identify a pair of twin primes that are both greater than 100.
4. Show that 3, 5, 7 is the only set of "**triplet primes**."
5. Either find all pairs of twin primes whose sum is a multiple of 3, or prove there are infinitely many.

Divisors – The Basics

1. Find the smallest prime divisor of $3^{456} + 5^{654}$?
2. All five geometric means are inserted between 8 and 5832, find the fifth term in the geometric series. *AHSME*

   Solution 1: Prime factorizations make the progression easy to spot:

   $$2^3,\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ \underline{\quad},\ 2^3 \cdot 3^6.$$

   The exponents of the primes must be in arithmetic progression (make sure you see why). So, we have

   $$2^3,\ \ 2^3 \cdot 3^1,\ \ 2^3 \cdot 3^2,\ \ 2^3 \cdot 3^3,\ \ 2^3 \cdot 3^4,\ \ 2^3 \cdot 3^5,\ \ 2^3 \cdot 3^6.$$

   The fifth term is $2^3 \cdot 3^4 = 648$.

   Solution 2: In the geometric progression, we are going 4 terms "up" from 8, and 2 terms "down" from 5832, which gives the term

   $$\sqrt[2+4]{8^2 \cdot 5832^4} = \sqrt[6]{\left(2^3\right)^2 \left(2^3 \cdot 3^6\right)^4} = \sqrt[6]{2^{18} \cdot 3^{24}} = 2^3 \cdot 3^4 = 648.$$

3. Find the smallest natural number that is not a divisor of 10080.
4. Let n be the smallest nonprime integer greater than 1 with no prime factor less than 10. Then *AHSME*

   (A) $100 < n \leq 110$
   (B) $110 < n \leq 120$
   (C) $120 < n \leq 130$
   (D) $130 < n \leq 140$

   (E) $140 < n \leq 150$

5. Find each of the following:
   (a) The prime factorization of 168.
   (b) Describe the prime factorization of any positive divisor of 168.
   (c) The positive divisors of 168.

Divisor Counting

Many students know the formula for counting the divisors of a natural number. However, more difficult problems can only be solved by those with a full understanding of the formula – how it is derived. Deriving the formula for the number of divisors of a natural number is an exercise in *organization* and *generalization*.

1. (a) Find the prime factorization of 180.
   (b) Describe the general prime factorization for a positive divisor of 180.
   (c) Find the total number of positive divisors of 180.

   Solution: First, we note that

$$180 = 2^2 \cdot 3^2 \cdot 5^1.$$

   Now, we note that any divisor of 180 must have a prime factorization of the form

$$2^a \cdot 3^b \cdot 5^c,$$

   where $0 \leq a \leq 2$, $0 \leq b \leq 2$, and $0 \leq c \leq 1$. When we divide 180 by an integer with any other prime factorization, there will be at least one prime in the denominator that doesn't cancel with the numerator resulting in a non-integer fraction.

   Finally, we take an organized look at the divisors of 180 according to our description of these divisors:

$$\begin{array}{rrr}
2^0 \cdot 3^0 \cdot 5^0 = 1 & 2^1 \cdot 3^0 \cdot 5^0 = 2 & 2^2 \cdot 3^0 \cdot 5^0 = 4 \\
2^0 \cdot 3^1 \cdot 5^0 = 3 & 2^1 \cdot 3^1 \cdot 5^0 = 6 & 2^2 \cdot 3^1 \cdot 5^0 = 12 \\
2^0 \cdot 3^2 \cdot 5^0 = 9 & 2^1 \cdot 3^2 \cdot 5^0 = 18 & 2^2 \cdot 3^2 \cdot 5^0 = 36 \\
\\
2^0 \cdot 3^0 \cdot 5^1 = 5 & 2^1 \cdot 3^0 \cdot 5^1 = 10 & 2^2 \cdot 3^0 \cdot 5^1 = 20 \\
2^0 \cdot 3^1 \cdot 5^1 = 15 & 2^1 \cdot 3^1 \cdot 5^1 = 30 & 2^2 \cdot 3^1 \cdot 5^1 = 60 \\
2^0 \cdot 3^2 \cdot 5^1 = 45 & 2^1 \cdot 3^2 \cdot 5^1 = 90 & 2^2 \cdot 3^2 \cdot 5^1 = 180
\end{array}$$

   - The maximum value of a is 2, and there are $2 + 1 = 3$ possible values for a.
   - The maximum value of b is 2, and there are $2 + 1 = 3$ possible values for b.
   - The maximum value of c is 1, and there are $1 + 1 = 2$ possible values for c.

   In total, there are

$$(2 + 1)(2 + 1)(1 + 1) = 3 \cdot 3 \cdot 2 = 18$$

   positive divisors of 180.

2. (a) Find the GCD of 600 and 720.
   (b) Find the prime factorization of your answer from (a).
   (c) Find the total number of positive integers that are common divisors of 600 and 720.

3. How many ordered pairs of integers (positive or negative) are there to the equation

$$xy = 120?$$

4. Find the smallest positive integer with exactly 9 positive divisors.

5. Find the smallest positive integer with exactly 8 divisors that are perfect squares.

6. In this problem we find the smallest positive integer that has exactly 12 positive divisors.
   (a) Find the least and the greatest possible number of primes that divide such an integer.
   (b) Find the smallest positive integer that has exactly 12 positive divisors.

7. How many natural numbers n less than 100 have exactly one divisor other than 1 and n? *Mandelbrot*

8. Find the product of the positive divisors of 600.

9. Find the product of the positive divisors of 2500 that are multiples of 10.

Divisors – More Advanced Problems

1. While working with some data for the Iowa City Hospital, James got up to get a drink of water. When he returned, his computer displayed the "blue screen of death" (it had crashed). While rebooting his computer, James remembered that he was nearly done with his calculations since the last time he saved his data. He also kicked himself for not saving before he got up from his desk. He had computed three positive integers a, b, and c, and recalled that their product is 24, but he didn't remember the values of the three integers themselves. What he really needed was their sum. He knows that the sum is an even two-digit integer less than 25 with fewer than 6 divisors. Help James by computing $a + b + c$. *iTest*

   Answer: 10

   Solution: We could try listing all possible ordered triples (a, b, c). However, even an organized approach might take a long time, particularly listing possible permutations such as

$$(12, 2, 1), (12, 1, 2), (2, 12, 1), (2, 1, 12), (1, 12, 2), (1, 2, 12).$$

   We can work much faster by ignoring the distinction between these different ordered triples. So, we only consider cases where a is at least as large as b, which is in turn at least as large as c.

   The mathematically technical way of saying this is "Without loss of generality, we let $a \geq b \geq c$."

   Now, since a is a divisor of 24, there are only a few possible cases:

- $a = 24$, in which case $bc = 1$, so $b = c = 1$. In this case $a + b + c = 26$.
- $a = 12$, in which case $bc = 2$, so $b = 2$ and $c = 1$. In this case $a + b + c = 15$.
- $a = 8$, in which case $bc = 3$, so $b = 3$ and $c = 1$. In this case $a + b + c = 12$.
- $a = 6$, in which case $bc = 4$. When $b = 4$ and $c = 1$, then $a+b+c = 11$. When $b = c = 2$, then $a + b + c = 10$.
- $a = 4$, in which case $bc = 6$, so $b = 3$ and $c = 2$. In this case $a + b + c = 9$.
- If $a \le 3$, then $bc \ge 8$, so the greater of b and c is at least $\sqrt{8} = 2\sqrt{2}$. Since

  $$4 < 8 < 9 \qquad \Leftrightarrow \qquad \sqrt{4} < \sqrt{8} < \sqrt{9},$$

  so $2 < 2\sqrt{2} < 3$. Since b and c are integers, the greater of the two is at least 3. But $3 \nmid 8$, so the greater of b and c is at least 4. This contradicts the fact that $a \ge b \ge c$, so we are done with our casework.

The possible even integer sums are 26, 12, and 10. Only 12 and 10 are less than 25, and 12 has six divisors, while 10 has four, so $a + b + c = 10$.

2. Let a/b be the probability that a randomly selected divisor of 2007 is a multiple of 3. If a and b are relatively prime positive integers, find $a + b$. *iTest*

   Answer: 5

   Solution 1: Prime factorization tends to be the foundation of most problems involving divisors, so we start there:

   $$2007 = 3^2 \cdot 223^1.$$

   A divisor of 2007 is of the form

   $$\pm 3^a \cdot 223^b,$$

   where $0 \le a \le 2$ and $0 \le b \le 1$.

   Now, we compute the total number of divisors by multiplying 2 choices for the signs, 3 choices for a, and 2 choices for b to get $2 \cdot 3 \cdot 2 = 12$. The multiples of 3 are the ones in which $a > 0$, and there are $2 \cdot 2 \cdot 2 = 8$. So, the probability that a randomly selected divisor of 2007 is a multiple of 3 is $8/12 = 2/3$, and the answer is $2 + 3 = 5$.

   Note: the answer is the same whether or not negative divisors are taken into account.

   Solution 2: Again, the divisors are of the form

   $$\pm 3^a \cdot 223^b.$$

   Since the values of a and b are *independent*, we only need to think about the values of a, which must be at least 1 in order for a divisor to be a multiple of 3. There are 3 possible values of a, of which 2 are at least 1, so the probability is $2/3$, and again the answer is $2 + 3 = 5$.

3. Let a/b be the probability that a randomly chosen positive divisor of 12^{2007} is also a divisor of 12^{2000}, where a and b are relatively prime positive integers. Find the remainder when $a+b$ is divided by 2007. *iTest TOC*

   Answer: 79

Solution: The number of positive divisors of

$$12^{2007} = 2^{4014} \cdot 3^{2007}$$

is $4015 \cdot 2008$. The number of positive divisors of

$$12^{2000} = 2^{4000} \cdot 3^{2000}$$

is $4001 \cdot 2001$, and all these are divisors of 12^{2007}. Thus, the probability that a positive divisor of 12^{2007} is also a divisor of 12^{2000} is equal to

$$\frac{4001 \cdot 2001}{4015 \cdot 2008}.$$

Now, in order to see whether or not this fraction reduces, we note each of the following:

$$\begin{aligned}
\gcd(4015, 4001) &= \gcd(4015, 14) &&= \gcd(11, 14) &&= 1,\\
\gcd(4015, 2001) &= \gcd(4015, 13) &&= \gcd(11, 13) &&= 1,\\
\gcd(2008, 4001) &= \gcd(2008, -15) &&= \gcd(13, -15) &&= 1,\\
\gcd(2008, 2001) &= \gcd(7, 2001) &&= \gcd(7, 6) &&= 1.
\end{aligned}$$

Thus, when we multiply out the numerator and denominator of our fraction above (the probability), the resulting fraction does not reduce. So, $a = 4015 \cdot 2008$ and $b = 4001 \cdot 2001$.

Now we compute

$$\begin{aligned}
4015 \cdot 2008 + 4001 \cdot 2001 &\equiv 1 \cdot 1 + (-13) \cdot (-6)\\
&\equiv 1 + 78 \equiv 79 \pmod{2007}.
\end{aligned}$$

Divisor Sums

Look back at the way you organized divisors in order to count them. Does that organization help you spot a nice way to simplify their sum, then simplify it again – possibly several times?

1. (a) Find the sum of the positive divisors of 8.
   (b) Find the sum of the positive divisors of 9.
   (c) Find the sum of the positive divisors of 72.
   (d) Find a general method for finding the sum of the positive divisors of an integer.

The *Structure* of Our Decimal Numerals

1. What is the sum of the digits of the decimal form of the product $2^{1999} \cdot 5^{2001}$?

   (A) 2 (B) 4 (C) 5 (D) 7 (E) 10

AHSME

Solution: We combine as many powers of 2 and 5 together to make powers of 10:

$$2^{1999} \cdot 5^{2001} = \left(2^{1999} \cdot 5^{1999}\right) \cdot 5^2 = 10^{1999} \cdot 25 = 2500000 \cdots 00000,$$

where the sum of the digits is $2 + 5 + 0 + 0 + 0 + \cdots + 0 + 0 = 7$.

2. The number of digits in $4^{16}5^{25}$ (when written in the usual base 10 form) is

(A) 31 (B) 30 (C) 29 (D) 28 (E) 27

AHSME

Base Numbers – The Basics

1. Write each of the following as a base 5 integer:
   (a) $2 \cdot 5^2 + 3 \cdot 5^1 + 0 \cdot 5^0$
   (b) $4 \cdot 5^3 + 3 \cdot 5^0$
   (c) $4 \cdot 5^3 + 1 \cdot 5^2 + 0 \cdot 5^1 + 1 \cdot 5^0$
   (d) $3 \cdot 5^5 + 1 \cdot 5^4 + 4 \cdot 5^3 + 2 \cdot 5^2 + 3 \cdot 5^1 + 4 \cdot 5^0$

Solutions: Given an integer equal to

$$c_n5^n + c_{n-1}5^{n-1} + c_{n-2}5^{n-2} + \cdots + c_25^2 + c_15^1 + c_05^0,$$

where $0 \le c_i \le 4$, its $(n+1)$-digit base 5 representation is equal to

$$c_nc_{n-1}c_{n-2} \cdots c_2c_1c_0,$$

where the c_i's are the $n+1$ digits.

(a)	230_5	(c)	4101_5
(b)	4003_5	(d)	314234_5

Make sure you see why (b) is a four-digit numeral and not a two-digit numeral.

2. Convert each of the following to base 10:

(a)	71_8	(c)	$2A2_{11}$	(e)	2144_6
(b)	133_9	(d)	4040_8	(f)	1020102_3

Solutions: Here I present solutions to some of the above conversions to display certain principles:

(a) $71_8 = 7 \cdot 8^1 + 1 \cdot 8^0 = 7 \cdot 8 + 1 \cdot 1 = 56 + 1 = 57$.

(c) The digit A has the value of 10 for bases higher than base 10:

$$2A2_{11} = 2 \cdot 11^2 + 10 \cdot 11^1 + 2 \cdot 11^0 = 242 + 110 + 2 = 354.$$

(d) We factor here for easier computation:

$$\begin{aligned} 4040_8 &= 4 \cdot 8^3 + 0 \cdot 8^2 + 4 \cdot 8^1 + 0 \cdot 8^0 \\ &= 4 \cdot 8^3 + 4 \cdot 8^1 \\ &= 4 \cdot 8^1(8^2 + 1) \\ &= 32(64 + 1) = 32 \cdot 65 = 16 \cdot 130 = 2080. \end{aligned}$$

3. Convert 125 to each of the following number bases:

   (a) 3
   (b) 4
   (c) 5
   (d) 6

4. Rewrite each of the following as a binary integer. Explain any general relationship.

   (a) 5
   (b) 10
   (c) 20
   (d) 40
   (e) 29
   (f) 58

5. How many three-digit numbers in base four do not have zero as their units digit?

6. How many three-digit base four numerals *do not* contain the digit 3?

7. In how many zeros does the number 77! end when written in base 12? What about in base 15?

8. How many integers from 1 to 1000 inclusive have a base 4 representation that does not include the digit 3?

9. For now many natural numbers N is 642_N a prime number?

10. If $ab_7 = ba_{13}$, then find the base 10 fraction $\frac{a}{b}$.

    Solution: An understanding of base numbers leads us to simple algebra:

    $$ab_7 = ba_{13} \qquad \Leftrightarrow \qquad 7a + b = 13b + a.$$

    Our equation simplifies to

    $$6a = 12b \qquad \Leftrightarrow \qquad \frac{a}{b} = \frac{12}{6} = 2.$$

11. Assume the letters x and y represent digits. Find (x, y) given that

    $$66x1_7 = 32y5_9.$$

12. The binary number N consists of 2007 digits, all of which are 1's. Find the number of 1's used when $3N$ is written in binary form.

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Base Numbers Revisited

1. The base three representation of x is

$$12112211122211112222.$$

The first digit (on the left) of the base nine representation of x is *AHSME*

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

2. The number 695 is to be written with a factorial base of numeration, that is,

$$695 = a_1 + a_2 \cdot 2! + a_3 \cdot 3! + \cdots + a_n \cdot n!$$

where a_1, a_2, $\ldots$, a_n are integers such that $0 \le a_k \le k$. Find a_4. *AHSME*

3. Consider the equation

$$100\cdots00_b + 100\cdots00_{b+1} = 100\cdots00_{b+2},$$

where each term contains exactly n zeros [note that each subscript indicates the base in which that term is written]. For how many values of n, $2 \le n \le 100$, will a solution [that is, a positive integer value of b] exist for the equation? *ARML*

4. If the base 8 representation of a perfect square is $ab3c$, where $a \ne 0$, then c is *AHSME*

(A) 0 (B) 1 (C) 3 (D) 4 (E) not uniquely determined

5. For each positive integer n, let $g(n)$ be the sum of the digits when n is written in binary. For how many positive integers n, where $1 \le n \le 2007$, is $g(n) \ge 3$? *iTest TOC*

Answer: 1941

Solution: We approach this problem using complementary counting. It's easier to count the integers n from 1 to 2007 inclusive such that $g(n) < 3$. We have just the cases $g(n) = 1$ and $g(n) = 2$.

- If $g(n) = 1$, then n has a binary representation that is a 1 followed by some number of zeros. In other words, n is a pure power of 2. Thus, $g(n) = 1$ for $2^0 = 1, 2^1 = 2, 2^2 = 4, \ldots, 2^9 = 512, 2^{10} = 1024$, which is 11 values.
- If $g(n) = 2$, then n is the sum of two distinct pure powers of 2. While counting these, we note that $2^{10} < 2007 < 2^{11}$ and also that $2^9 + 2^{10} < 2007$. This means $g(n) = 2$ for exactly the integers $n = 2^a + 2^b$, where (a, b) are distinct nonnegative integers no greater than 10, and $a > b$. There are $\binom{11}{2} = 55$ such integers n.

In total, there are $11 + 55 = 66$ integers n such that $1 \le n \le 2007$ and $g(n) < 3$. This leaves $2007 - 66 = 1941$ integers in that range such that $g(n) \ge 3$.

Remainder Math a.k.a. Modular Arithmetic

The **Division Theorem**, which is sometimes referred to as the **Division Algorithm**, states that for any integer a and positive integer b, there exists exactly one pair of integers q and r such that

$$a = bq + r,$$

where $0 \leq r < b$. Here, a is the **dividend** which gets divided by b, is the **divisor**. The resulting **quotient** is q, and the remainder is r.

Unfortunately, many students never learn about the Division Theorem. While seemingly trivial, getting accustomed to thinking about integers expressed in the form above not only opens up avenues to solving many problems, but serves as the basis for a study of modular arithmetic – one of the most useful tools in all of mathematics!

1. The odd positive integers, 1, 3, 5, 7, . . . , are arranged in five columns continuing with the pattern shown on the right. Counting from the left, the column in which 1985 appears is the

	1	3	5	7
15	13	11	9	
	17	19	21	23
31	29	27	25	
	33	35	37	39
47	45	43	41	
	49	51	53	55
*	*	*	*	
	*	*	*	*
*	*	*	*	
	*	*	*	*

   (A) first

   (B) second

   (C) third

   (D) fourth

   (E) fifth

   AHSME

   Answer: (B) second

   Solution: Looking for regularity, we note that the list repeats its form every other row. So, every two rows a total of $2 \cdot 4 = 8$ odd numbers appear. This means that the numbers increase by $2 \cdot 8 = 16$ every other row.

   A pattern of numbers that increase by 16 is an arithmetic progression. Consider the progression in the fourth column of the odd-numbered rows: 5, 21, 37, 53, ... These numbers are of the form $16n + 5$ for nonnegative integers n. This means they all *share the same remainder.*

   In general, this observation allows us to use remainders to identify the column in which each positive odd integer lies. For instance,

$$1985 = 124 \cdot 16 + 1,$$

   where 1 is the remainder when 1985 gets divided by 16. So, 1985 lies in the same column as 1, which is the second column.

2. What is the smallest positive four-digit number that gives a quotient of 219 with remainder 17 when divided by some positive one-digit number?

3. Sarah and Stacey play a game with 82 toothpicks. Starting with Sarah, the players take turns removing from 1 to 5 toothpicks (inclusive) from the pile. The player who must take the last toothpick loses. Help Sarah formulate a winning strategy.

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An Introduction to Modular Arithmetic

Modular arithmetic encompasses the mathematics of the remainders of the integers as they are expressed in the Division Algorithm. For the most part, modular arithmetic is a tool of computational efficiency. In computer science, it is the basis for much of the theory of Cryptography.

First, in modular arithmetic we define some new notation, and a relationship called **congruence** or **equivalence**. When we say that 53 is congruent to 13 modulo 10, we mean that $53 - 13 = 40$ is a multiple of 10. We write this as

$$53 \equiv 13 \pmod{10}.$$

This is another way of saying that 53 and 13 have the same remainder upon division by 10 in the division algorithm:

$$\begin{aligned} 13 &= 1 \cdot 10 + 3, \\ 53 &= 5 \cdot 10 + 3. \end{aligned}$$

However, instead of using the word "remainder", we say that 3 is the **residue** of 53 (and of 13) in modulo 10.

More generally, when $a \equiv b \pmod{m}$, then

$$\frac{a-b}{m}$$

is an integer. Then also, a and b share the same modulo m residue.

Now, note each of the following examples of congruence ($\equiv$) and lack of congruence ($\not\equiv$):

$$\begin{array}{lllll} \dfrac{88-4}{7} = \dfrac{84}{7} = 12, & \text{which is an integer, therefore,} & 88 \equiv 4 \pmod{7}, \\ \dfrac{88-5}{7} = \dfrac{83}{7} = 11\tfrac{6}{7}, & \text{which is } \textbf{not} \text{ an integer, therefore,} & 88 \not\equiv 5 \pmod{7}, \\ \dfrac{197-72}{25} = \dfrac{125}{25} = 5, & \text{which is an integer, therefore,} & 197 \equiv 72 \pmod{7}, \\ \dfrac{1907-72}{25} = \dfrac{1835}{25} = 73\tfrac{2}{5}, & \text{which is } \textbf{not} \text{ an integer, therefore,} & 1907 \not\equiv 72 \pmod{25}. \end{array}$$

In modular arithmetic, the integers are partitioned into categories called **residue classes** based on their remainders as expressed in the Division Theorem. For instance,

$$\begin{aligned} &\vdots \\ -10 &= -2 \cdot 7 + 4, \\ -3 &= -1 \cdot 7 + 4, \\ 4 &= 0 \cdot 7 + 4, \\ 11 &= 1 \cdot 7 + 4, \\ 18 &= 2 \cdot 7 + 4, \\ 25 &= 3 \cdot 7 + 4, \\ 32 &= 4 \cdot 7 + 4, \\ &\vdots \end{aligned}$$

So, all these integers are in the same modulo 7 residue class. They each have 4 as their modulo 7 residue, and so they are equivalent in modulo 7:

$$-10 \equiv -3 \equiv 4 \equiv 11 \equiv 18 \equiv 25 \equiv 32 \pmod{7}.$$

Note that residue classes are sometimes called **congruence classes** or **equivalence classes**.

1. Prove each of the following laws of modular arithmetic, where n is a natural number and

$$\begin{aligned} x_1 &\equiv x_2 \pmod{m}, \\ y_1 &\equiv y_2 \pmod{m}. \end{aligned}$$

   (a) If $x_1 \equiv x_2 \pmod{m}$, and $x_2 \equiv x_3 \pmod{m}$, then

$$x_1 \equiv x_3 \pmod{m}.$$

   (b) $x_1 + y_1 \equiv x_2 + y_2 \pmod{m}$.
   (c) $x_1 - y_1 \equiv x_2 - y_2 \pmod{m}$.
   (d) $x_1 y_1 \equiv x_2 y_2 \pmod{m}$.
   (e) $x_1^n \equiv x_2^n \pmod{m}$.

Solution to (a): By definition, we know that

$$\frac{x_1 - x_2}{m} = k_1,$$

where k_1 is an integer, and also

$$\frac{x_2 - x_3}{m} = k_2,$$

where k_2 is an integer. Thus,

$$\frac{x_1 - x_2}{m} - \frac{x_2 - x_3}{m} = k_1 - k_2 \qquad \Leftrightarrow \qquad \frac{x_1 - x_3}{m} = k_1 - k_2.$$

Since k_1 and k_2 are integers, which are closed under subtraction, $\frac{x_1 - x_3}{m}$ is an integer, thus $x_1 \equiv x_3 \pmod{m}$.

Solution to (d): From the given congruences, we can say that

$$\begin{aligned} x_1 &= q_1 m + r_1, \\ x_2 &= q_2 m + r_1, \\ y_1 &= q_3 m + r_2, \\ y_2 &= q_4 m + r_2. \end{aligned}$$

So,

$$x_1 y_1 = (q_1 m + r_1)(q_3 m + r_2) = (q_1 q_2 m + q_1 r_2 + q_3 r_1)m + r_1 r_2 \equiv r_1 r_2 \pmod{m}.$$

Similarly, $x_2 y_2 \equiv r_1 r_2 \pmod{m}$. Then by transitivity (as we proved in (a)), we have that $x_1 y_1 \equiv x_2 y_2 \pmod{m}$.

2. Let S be equal to the sum
$$1 + 2 + 3 + \cdots + 2007.$$
Find the remainder when S is divided by 1000. *iTest*

Answer: 28

Solution 1: The sum of the first 2007 positive integers, like any arithmetic series, can be computed using a "copy" of the series, reversed:
$$\begin{array}{rcccccccc} S = & 1 & + & 2 & + & 3 & + \cdots + & 2007, \\ S = & 2007 & + & 2006 & + & 2005 & + \cdots + & 1, \\ S + S = & (1+2007) & + & (2+2006) & + & (3+2005) & + \cdots + & (2007+1). \end{array}$$
Adding the first two of the above three equations gave us the third, which is in fact the sum when the number 2008 is added together a total of 2007 times:
$$2S = 2008 + 2008 + 2008 + \cdots + 2008 = 2007 \cdot 2008 = 4030056.$$
Thus, $S = 2015028$, which leaves a remainder of 28 when divided by 1000.

Solution 2: Modular arithmetic allows for simplified computation of the remainder:
$$S = \frac{2007 \cdot 2008}{2} = 2007 \cdot 1004 \equiv 7 \cdot 4 \equiv 28 \pmod{1000}.$$
Solution 3: Some clever students, not knowing a formula for the sum of an arithmetic series, or the modular arithmetic in Solution 2, might pair terms up with no remainder:
$$(1 + 1999) + (2 + 1998) + (3 + 1997) + \cdots + (999 + 1001)$$
$$+1000 + 2000 + 2001 + 2002 + \cdots + 2007.$$
Casting out the multiples of 1000, we have the simple sum $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$.

3. Find the remainder when $4511 \cdot 17^{2007}$ is divided by 6.

4. What is the remainder when 2^{1994} is divided by 17.
Solution: We note that $2^4 = 16 \equiv -1 \pmod{17}$, and so
$$2^8 = (2^4)^2 \equiv (-1)^2 \equiv 1 \pmod{17}.$$
We make use of this identity:
$$2^{1994} = (2^8)^{124} \cdot 2^2 \equiv (1)^{124} \cdot 4 \equiv 4 \pmod{17},$$
so 4 is the remainder.

5. Prove that if $x \equiv 11 \pmod{24}$, then $3x \equiv 1 \pmod{8}$.

6. Find a where
$$11 \cdot 18 \cdot 2322 \cdot 13 \cdot 19 \equiv a \pmod{7}.$$

7. Which one of the following integers can be expressed as the sum of 100 consecutive positive integers?

   (A) 1,627,384,950 (B) 2,345,678,910 (C) 3,579,111,300

   (D) 4,692,581,470 (E) 5,815,937,260

   AHSME

8. Consider the "tower of power" $2^{2^{\cdot^{\cdot^{2}}}}$, where there are 2007 twos including the base. What is the last (units) digit of this number? *iTest*

   Answer: 6

   Solution: Note that the last digit of powers of 2 repeats every fourth term (2, 4, 8, 6, 2, 4, 8, 6, . . .). The number to which we raise 2 here is obviously a multiple of 4, so the last digit of the expression will be 6.

9. Find the number of elements in the largest subset, S, of the set $\{1, 2, 3, \ldots, 2007\}$ such that no pair of distinct elements of S has a sum divisible by 7.

10. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. Then the number of real solutions of $4x^2 - 40\lfloor x \rfloor + 51 = 0$ is

    (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

    AHSME

11. Show that $17^{2007} + 22^{2007}$ is a multiple of 13.

Units Digits

Since the units digit of an integer gives us its modulo 10 residue, and vice versa (so long as we know the sign of the integer), we can apply modular arithmetic to units digit problems.

1. Find the units digit of each of the following:

   (a) $3912567 + 7439214$ (c) $49741 \cdot 73973$

   (b) $41 \cdot 73$ (d) 4^{13}

2. The units digit of $3^{1001}7^{1002}13^{1003}$ is

   (A) 1 (B) 3 (C) 5 (D) 7 (E) 9

   AHSME

3. Find the units digit of the sum

$$1! + 2! + 3! + \cdots + 2007!.$$

4. Find the units digit of the sum

$$(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2 + \cdots + (2007!)^2.$$

   iTest

   Answer: 7

   Solution: The units digit of any multiple of 10 is 0. After the first four terms of the given series, all the terms have units digit 0. Consider,

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1,$$

   is a multiple of $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ for all $n \geq 5$. So, $n!$ is a multiple of 10, and therefore, so is $(n!)^2$.

   The units digit of the sum of a group of integers is the sum of those integers' units digits. So, first, we compute

$$\begin{aligned}(1!)^2 &= 1^2 = 1,\\ (2!)^2 &= 2^2 = 4,\\ (3!)^2 &= 6^2 = 36,\\ (4!)^2 &= 24^2 = 576.\end{aligned}$$

   The rest of the terms have units digit 0, so we are looking for the units digit of

$$1 + 4 + 36 + 576 + 0 + 0 + \cdots + 0 + 0 = 617,$$

   which is 7.

5. The cube of the three-digit natural number $A7B$ is 108531333. What is $A + B$? *MATHCOUNTS*

6. What is the units digit of $13^{242} + 17^{378}$?

7. Find all possible units digits of a perfect square.

8. How many three-digit integers are multiples of both 4 and 6 and have a units digit of 8?

9. The tens digit of a perfect square is odd. What is the units digit?

More Digits

1. Find the tens digit of 3^{2007}.

2. Find the last two digits of 2007^{2007}. Can you find the hundreds digit?

Divisibility Rules

1. Show that every natural number is

   (a) congruent to its units digit modulo 2,
   (b) congruent to its units digit modulo 5, and
   (c) congruent to its units digit modulo 10.

2. A four-digit number uses each of the digits 1, 2, 3 and 4 exactly once. What is the probability that the number is a multiple of 4? *MATHCOUNTS*

3. Prove that for any positive integer m, every integer is congruent to its last m digits in modulo 10^m.

4. Show that every natural number is
   (a) congruent to its digit sum modulo 3,
   (b) congruent to its digit sum modulo 9,
   (c) congruent to its alternating digit sum modulo 11.

   Solution to (a): Let n be a natural number and let $d_k, d_{k-1}, d_{k-2}, \ldots, d_1, d_0$ be its digits from left to right. Then

$$\begin{aligned} n &= d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + d_{k-2} \cdot 10^{k-2} + \cdots + d_1 \cdot 10^1 + d_0 \cdot 10^0 \\ &\equiv d_k \cdot 1^k + d_{k-1} \cdot 1^{k-1} + d_{k-2} \cdot 1^{k-2} + \cdots + d_1 \cdot 1^1 + d_0 \cdot 1^0 \\ &\equiv d_k \cdot 1 + d_{k-1} \cdot 1 + d_{k-2} \cdot 1 + \cdots + d_1 \cdot 1 + d_0 \cdot 1 \\ &\equiv d_k + d_{k-1} + d_{k-2} + \cdots + d_1 + d_0 \pmod{3}, \end{aligned}$$

   which is to say that an integer is congruent in modulo 3 to the sum of its digits.

   Note 1: The above proof is a consequence of the fact that $10 \equiv 1 \pmod{3}$. In other words, the base of the decimal system is a modulo 3 *unit*.

   Note 2: Recognize that the divisibility rule for 3 is the special case where $3 \mid n$, and the sum of the digits is also a multiple of 3.

5. Let A and B represent digits of the decimal number $40A5B$, which is a multiple of 72. Find the ordered pair (A, B).

6. (a) If the number $A3640548981270644B$ is divisible by 99, compute the ordered pair (A, B). *NYSML*
   (b) Find a divisibility rule for 99 and demonstrate that it works.
   (c) Does (b) give a faster solution to (a)?

7. Dr. Erdos has three dials: the leftmost contains the digits 1 and 2, the middle shows the digits 0, 4, and 8, and the rightmost has the digits 3, 5, 6, and 7. How many three digit prime numbers can he create using the dials? *Mandelbrot*

8. The letters A, B, and C represent different digits, A is prime, and $A - B = 4$. If the number $AAABBBC$ is a prime, compute the ordered triple (A, B, C). *ARML*

9. Find the smallest natural number, all of whose digits are 1, that is a multiple of 333.

10. The sum of the digits of an integer is equal to the sum of the digits of three times that integer. Prove that the integer is a multiple of 9. *iTest*

    Solution: Let $S(n)$ be the sum of the digits of an integer n. Now we can write the given information as an equation: $S(n) = S(3n)$. It is also true that an integer is congruent to the sum of its digit modulo 9 (try to prove this fact if you have not before). Thus,

    $$\begin{aligned} S(n) &\equiv n \pmod{9}, \\ S(3n) &\equiv 3n \pmod{9}. \end{aligned}$$

    Now, subtracting the first equation from the second, we get

    $$S(3n) - S(n) \equiv 3n - n \pmod{9}.$$

    Since $S(n) = S(3n)$, we have that $S(3n) - S(n) = 0$, and so

    $$S(3n) - S(n) = 0 \equiv 2n \pmod{9} \qquad \Rightarrow \qquad 0 \equiv n \pmod{9},$$

    which is to say that n is a multiple of 9.

    Follow Up Problem: For which integers k is it always true that

    $$S(n) = S(kn) \qquad \Rightarrow \qquad n \equiv 0 \pmod{9}?$$

Linear Congruences

Just as we solve for unknowns using arithmetic manipulations of equations, we apply algebra similarly to modular arithmetic problems. Here we take a look at problems involving variables of degree 1, such as

$$3n \equiv 1 \pmod{13}.$$

Solving this **linear congruence** is more difficult than solving a linear equation, but once we learn the process, we can solve them without tremendous difficulty.

1. Find the sum of all integers n such that $100 \le n \le 200$ and

   $$3n \equiv 1 \pmod{13}.$$

2. Solve: $4x \equiv 24 \pmod{28}$.

Systems of Linear Congruence

1. Find all solutions to the following system of linear congruences:

   $$\begin{aligned} n &\equiv 1 \pmod{3}, \\ n &\equiv 2 \pmod{4}. \end{aligned}$$

2. Find all solutions to the following system of linear congruences:

$$\begin{aligned} n &\equiv 3 \pmod{4}, \\ n &\equiv 2 \pmod{6}. \end{aligned}$$

3. How many three-digit positive integers leave a remainder of 3 when divided by 5, and a remainder of 2 when divided by 3?

4. How many positive multiples of 9 less than 100 leave a remainder of 4 when divided by 7?

5. Find the sum of all prime numbers between 1 and 100 that are simultaneously 1 greater than a multiple of 4 and 1 less than a multiple of 5.

   (A) 118 (B) 137 (C) 158 (D) 187 (E) 245

   AHSME

6. Adam and Ben start their new jobs on the same day. Adam's schedule is 3 workdays followed by 1 rest day. Ben's schedule is 7 workdays followed by 3 rest days. On how many of their first 1000 days do both have rest days on the same day? *AMC*

7. Find all solutions to the following system of linear congruences:

$$\begin{aligned} n &\equiv 1 \pmod{2}, \\ n &\equiv 2 \pmod{3}, \\ n &\equiv 3 \pmod{5}. \end{aligned}$$

8. Anna has more than 10 marbles, and fewer than 40. When she sorts her marbles into piles of 5, she finds that there are 2 marbles left over. When she sorts them into piles of 6, there are again 2 marbles left over. How many marbles does Anna have?

9. Find the smallest counting number that leaves a remainder of 3 when divided by 5, a remainder of 1 when divided by 3, and a remainder of 1 when divided by 2. (Recall that a counting number is a positive integer.)

10. Note that there are solutions to each of the following linear congruences:

$$\begin{aligned} 5x &\equiv 1 \pmod{18}, \\ 7x &\equiv 1 \pmod{18}. \end{aligned}$$

    For which integers k, such that $1 \leq k \leq 17$, are there solutions to the linear congruence

$$kx \equiv 1 \pmod{18}?$$

    Note: A solution to $kx \equiv 1 \pmod{18}$ is known as the modulo 18 inverse of k, and is unique if it exists.

11. Prove that for any positive integer $n > 1$, that $n - 1$ is its own inverse modulo n.

12. If three times a positive integer leaves a remainder of one when divided by four, four times the integer leaves a remainder of 5 when divided by 7, and five times the integer leaves a remainder of 8 when divided by 9, what is the smallest possible value of the integer?
13. Show that in any modulus, no integer has more than one inverse (at least, all integers that satisfy the inverse equation are congruent in that modulus).

Properties of Squares and Other Algebraic Forms

1. Let a and b be positive integers. Prove that if $a^2 + b^2$ is a multiple of 3, then *both* a and b are also multiples of 3.
2. A natural number n has a units digit of A when expressed in base 12. Find the remainder when n^2 is divided by 12.
3. There is a 2007-digit prime number p whose square *does not* leave a remainder of 1 when divided by 120. What is the remainder?
4. Denote by p_k the k^{th} prime number. Show that $p_1 p_2 p_3 \cdots p_n + 1$ cannot be the perfect square of an integer. *M& IQ*
5. Silas once had between 30 and 35 special coins, each marked with a different "value". These values were successive powers of 2, namely: $1, 2, 4, 8, \ldots$. Some of these coins were then lost. The total value of the lost coins was exactly 1/5 of the total value of all the original coins. Compute the number of lost coins. *ARML*
6. Determine all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ if any, apart from permutations, of the Diophantine equation

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1{,}599.$$

USAMO

Fermat's Little Theorem

Fermat's Little Theorem states that for a prime number p, and an integer n such that $\gcd(p, n) = 1$,

$$n^{p-1} \equiv 1 \pmod{p},$$

or equivalently,

$$n^p \equiv n \pmod{p}.$$

1. Find the remainder when 3^{201} gets divided by 11. *Mu Alpha Theta*

2. Find the remainder when 3^{2007} is divided by 2007. *iTest TOC*

   Answer: 1620

   Solution: Since $3^2 \cdot 223^1$ is the prime factorization of 2007, we can find the modulo 2007 residue of a number from its modulo 9 and modulo 223 residues.

   Clearly $3^{2007} \equiv 0 \pmod{9}$. By Fermat's Little Theorem, $3^{222} \equiv 1 \pmod{223}$. So,

$$\begin{aligned}
3^{2007} &\equiv \left(3^{222}\right)^9 \cdot 3^9 \equiv 1^9 \cdot 3^9 \\
&\equiv 3^9 \equiv 3^4 \cdot 3^5 \\
&\equiv 81 \cdot 243 \equiv 81 \cdot 20 \\
&\equiv 1620 \pmod{223}.
\end{aligned}$$

   Since $9 \mid 1620$, we have our answer. Otherwise, we could note that $1620 \equiv 59 \pmod{223}$ and solve the system of linear congruences

$$\begin{aligned}
r &\equiv 0 \pmod{9}, \\
r &\equiv 59 \pmod{223}.
\end{aligned}$$

Euler's Theorem (and Phi Function)

There is a theorem that generalizes Fermat's Little Theorem. But first, we define **Euler's Phi function**, which is also often called the **totient function**. We use the symbol ϕ to denote the Phi function, and the value of the function at a natural number n is defined at the number of positive integers no greater than n and relatively prime to n. For

$$n = p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots p_k^{e_k},$$

we apply the Principle of Inclusion-Exclusion (PIE) to find that

$$\begin{aligned}
\phi(n) &= \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) \cdots \left(1 - \frac{1}{p_k}\right) n \\
&= \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) \cdots \left(1 - \frac{1}{p_k}\right) p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \cdots p_k^{e_k} \\
&= (p_1 - 1)(p_2 - 1)(p_3 - 1) \cdots (p_k - 1) p_1^{e_1 - 1} p_2^{e_2 - 1} p_3^{e_3 - 1} \cdots p_k^{e_k - 1}.
\end{aligned}$$

Using the Phi function, we define **Euler's Theorem**, which states that for relatively prime natural numbers a and n,

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

For instance, $\phi(10) = 4$, and $\gcd(3, 10) = 1$, thus

$$3^{\phi(10)} = 3^4 \equiv 1 \pmod{10}.$$

1. Compute each of the following:

(a)	$\phi(10)$	(c)	$\phi(1440)$
(b)	$\phi(100)$	(d)	$\phi(2^{20})$

Solution for (a): $\phi(10) = (2-1)(5-1)2^{1-1}5^{1-1} = 1 \cdot 4 \cdot 1 \cdot 1 = 4$.

Solution for (c): $\phi(1440) = (2-1)(3-1)(5-1)2^{5-1}3^{2-1}5^{1-1} = 1 \cdot 2 \cdot 4 \cdot 16 \cdot 3 \cdot 1 = 384$.

2. Let $\phi(n)$ be the number of positive integers $k < n$ which are relatively prime to n. For how many distinct values of n is $\phi(n)$ equal to 12? *iTest*

   Answer: 6

   Solution: If $p_1, p_2, \ldots, p_k$ are the distinct prime factors of n, then $\phi(n) = n(1-1/p_1)(1-1/p_2)\cdots(1-1/p_k)$. Suppose that this equals 12. If n has no repeated prime factors, then $(p_1-1)(p_2-1)\cdots(p_k-1) = 12$. Check all possible factorizations of 12: $12, 1\cdot 12, 2\cdot 6, 1\cdot 2\cdot 6, 3\cdot 4, 1\cdot 3\cdot 4, 2\cdot 2\cdot 3, 1\cdot 2\cdot 2\cdot 3$. This gives $n = 13, 21, 26, 42$. If n has one repeated prime factor (repeated once), then $p_1(p_1-1)(p_2-1)\cdots(p_k-1) = 12$. This can be achieved with $n = 28$. If n has two repeated prime factors, then $p_1(p_1-1)p_2(p_2-1) = 12$. This can be satisfied by $n = 36$. It is easy to check that 12 doesn't have enough factors for any other possibility to work. So the answer is 6.

3. Find the remainder when 2007^{2008} gets divided by 2009.

4. Find the last two digits of 923^{1994}.

Wilson's Theorem

Wilson's Theorem states that for a prime number p,

$$(p-1)! \equiv -1 \pmod{p}.$$

1. Find the remainder when 309! is divided by 311.

2. Let a be the integer such that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{22} + \frac{1}{23} = \frac{a}{23!}$. Compute the remainder when a is divided by 13. *ARML*

Modular Arithmetic Revisited

1. Do there exist 21 consecutive positive integers each of which is divisible by one or more primes p from the interval $2 \le p \le 13$? *USAMO*

2. A sequence $a_1, a_2, a_3, \ldots$ is defined as follows: $a_1 = 2007$, and $a_n = a_{n-1} + n \pmod{k}$, where $0 \le a_n < k$. For how many values of k, where $2007 < k < 10^{12}$, does the sequence assume all k possible values (modulo k residues)? *iTest TOC*

   Answer: 29

Solution: We begin by noting that the given sequence takes on all modulo k residues if and only if the following sequence assumes all modulo k residues: $b_1 = 1$, and $b_n = b_{n-1} + n \pmod{k}$, where $0 \le b_n < k$. This is because $b_n \equiv a_n - 2006 \pmod{k}$, where $a_n - 2006$ ranges through all modulo k residues. This simplifies matters since

$$b_n \equiv 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \pmod{k}.$$

Now we have a closed form to work with. It's easy to see that this closed form is periodic modulo k:

$$b_{n+k} - b_n = \frac{(n+k)(n+k+1)}{2} - \frac{n(n+1)}{2} = nk + \frac{k(k+1)}{2} \equiv \frac{k(k+1)}{2} \pmod{k}.$$

When k is odd, $k(k+1)/2 = km \equiv 0 \pmod{k}$, where $m = (k+1)/2$ is an integer. Thus, when k is odd, the sequence has a period of k. Otherwise, k is even, and we can show similarly that $2k$ is a period.

In addition to periodicity, we find that the sequence exhibits symmetry. When k is odd,

$$b_{k-n-1} = \frac{(k-n)(k-n-1)}{2} = k\left(\frac{k-1}{2} - n\right) + \frac{n(n+1)}{2} \equiv \frac{n(n+1)}{2} \equiv b_n \pmod{k}.$$

The period of k implies that the sequence ranges through all modulo k residues if and only if $b_1, b_2, \ldots, b_k$ are distinct, but they are not. So we are only looking for even values of k.

When k is even, symmetry is still present within the period, but the period is $2k$:

$$b_{2k-n-1} = \frac{(2k-n-1)(2k-n)}{2} = \frac{(n+1-2k)(n-2k)}{2} \equiv \frac{n(n+1)}{2} \pmod{k}.$$

So, the sequence assumes all k possible values if and only if the first k terms assume all k possible values.

Plugging in possible even integers k, we find that $k = 2$, $k = 4$, and $k = 8$ work, while $k = 6$ and $k = 10$ do not. If we stop here and conjecture that powers of 2 work, we find further that $k = 12$ and $k = 14$ do not work, while $k = 16$ does.

First, we go about trying to prove that powers of 2 work. Supposing k is a power of 2, we note that $a_r = a_s$ for $1 \le r < s \le k$ iff

$$\frac{s(s+1)}{2} - \frac{r(r+1)}{2} = \frac{(s-r)(r+s+1)}{2} \equiv 0 \pmod{k} \quad \Rightarrow \quad (s-r)(r+s+1) \equiv 0 \pmod{2k}.$$

Both $s - r$ and $r + s + 1$ are less than $2k$, which is a power of 2. Thus, both $s - r$ and $r + s + 1$ are even. However, the parities of $s - r$ and $r + s + 1$ are different. Thus, it cannot be true that $a_r = a_s$, hence k can be any power of 2.

Now, note that if $p \mid k$ for some prime p, then

$$\frac{n(n+1)}{2} \equiv a \pmod{k} \quad \Rightarrow \quad \frac{n(n+1)}{2} \equiv a \pmod{p}.$$

So if k works, so does p. However, no odd numbers work, so $p = 2$ is the *only* prime divisor of k.

Finally, we tally the powers of 2 between 2007 and 10^{12}. First, we note that $2^{10} = 1024 < 2007 < 2048 = 2^{11}$, so 2^{11} is the smallest possible value of k. Next, we note that $2^{10} = 1024$, which is very close to 10^3. A bit of estimation and we find that $2^{39} < 10^{12} < 2^{40}$, so 2^{39} is the largest possible value of k. The number of possible values of k is thus $39 - 11 + 1 = 29$.

Algebraic Forms of Rational Numbers

Once you know a bit of number theory, throw in a solid background in algebra, and your problem solving potential explodes.

1. If the digit 1 is placed after a two digit number whose tens' digit is t, and units' digit is u, the new number is: *AHSME*

   (A) $10t + u + 1$ (B) $100t + 10u + 1$ (C) $1000t + 10u + 1$

   (D) $t + u + 1$ (E) none of these answers

2. How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square?

   (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

   AHSME

3. The square of an integer is called a *perfect square.* If x is a perfect square, the next larger perfect square is *AHSME*

   (A) $x + 1$ (B) $x^2 + 1$ (C) $x^2 + 2x + 1$ (D) $x^2 + x$ (E) $x + 2\sqrt{x} + 1$

4. Note the following:

$$\begin{array}{rcr}
0 \cdot 1 = 0 & \rightarrow & 5^2 = 25, \\
1 \cdot 2 = 2 & \rightarrow & 15^2 = 225, \\
2 \cdot 3 = 6 & \rightarrow & 25^2 = 625, \\
3 \cdot 4 = 12 & \rightarrow & 35^2 = 1225, \\
4 \cdot 5 = 20 & \rightarrow & 45^2 = 2025, \\
5 \cdot 6 = 30 & \rightarrow & 55^2 = 3025, \\
 & \vdots &
\end{array}$$

   Explain the relationship we see in squares of integers with units digit 5.

5. How many ordered pairs of positive integers (x, y), where $1993 < x < y < 2020$, satisfy the equation $y^2 - x^2 = 2x + 1$? *ARML*

6. Find the eighteenth digit in the decimal expansion of 1/97.

7. If $\frac{4}{2001} < \frac{a}{a+b} < \frac{5}{2001}$, compute the number of integral values that $\frac{b}{a}$ can take on. *ARML*
   Hint: In the middle fraction, it's the denominator that looks ugly. Find a way to make the fraction look prettier.

8. Given that a and b are natural numbers such that

$$\frac{11}{15} > \frac{a}{b} > \frac{7}{10},$$

   find the smallest possible value for b. *MATHCOUNTS*

   Hint: The answer is *not* 60. Find another way to look at all three numbers at once.

Making Use of Variables

Some problems don't appear to be algebra problems on face, but inserting variables where there are none can simplify the task of solving them significantly.

1. Find the smallest positive integer greater than 2 than leaves a remainder of 2 when divided by each 3, 4, 5, 6, and 7.

2. Compute
$$\frac{(1990)^3-(1000)^3-(990)^3}{(1990)(1000)(990)}.$$
ARML

   Solution: The repeated use of the numbers involved, along with their simple relationship,
$$990+1000=1990,$$
suggest that we might be able to examine the computations more generally. Letting $x=1000$ and $y=990$, the given fraction becomes
$$\frac{(x+y)^3-x^3-y^3}{(x+y)xy}=\frac{3x^2y+3xy^2}{(x+y)xy}=\frac{3xy(x+y)}{(x+y)xy}=3.$$
Assuming nonzero denominators, the given fraction evaluates to 3 regardless of the values of x and y.

3. If 1998 is written as a product of two positive integers whose difference is as small as possible, then the difference is

   (A) 8 (B) 15 (C) 17 (D) 47 (E) 93

   AHSME

4. Note that 10 is 9 more than the sum of the squares of its digits. Compute the sum of all other positive two-digit base 10 integers which are 9 more than the sum of the squares of their digits.
ARML

Factoring and Factoring

1. If $C=19!$, then express $21!-20!$ in terms of C.

2. If $2^{1998}-2^{1997}-2^{1996}+2^{1995}=k\cdot 2^{1995}$, what is the value of k?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

   AHSME

3. $\frac{1000^2}{252^2-248^2}$ equals

   (A) 62,500 (B) 1000 (C) 500 (D) 250 (E) $\frac{1}{2}$

4. Is 10609 prime?

   Solution: Letting $x = 100$, we note that

   $$10609 = x^2 + 6x + 9 = (x+3)^2 = 103^2,$$

   which is composite.

5. Compute the following:

   $$71^3 - 3 \cdot 71^2 + 3 \cdot 71^1 - 1 \cdot 71^0.$$

6. In their base 10 representations, the integer a consists of a sequence of 1985 eights and the integer b consists of a sequence of 1985 fives. What is the sum of the digits of the base 10 representation of the integer $9ab$?

   (A) 15880 (B) 17856 (C) 17865 (D) 17874 (E) 19851

   AHSME

7. Find the greatest integer that divides $k^3 - k$ for all positive integers k.

8. Find the largest prime divisor of $6^7 + 37$. *Mandelbrot*

9. The number of ordered pairs of integers (m, n) for which $mn \geq 0$ and

   $$m^3 + n^3 + 99mn = 33^3$$

   is equal to

   (A) 2 (B) 3 (C) 33 (D) 35 (E) 99

   AHSME

10. Find the largest integer n such that $2007^{1024} - 1$ is divisible by 2^n. *iTest*

    Answer: 13

    Solution: By repeated use of difference of squares factorization, we see that

    $$\begin{aligned} 2007^{1024} - 1 &= (2007^{512}+1)(2007^{512}-1) \\ &= (2007^{512}+1)(2007^{256}+1)(2007^{256}-1) \\ &\vdots \\ &= (2007^{512}+1)(2007^{256}+1)(2007^{128}+1)\cdots(2007+1)(2007-1). \end{aligned}$$

    Most of the factors are of the form $2007^{2^k} + 1$, and we note that for $k \geq 1$,

    $$2007^{2^k} \equiv (-1)^{2^k} \equiv 1^{2^{k-1}} \equiv 1 \pmod 4.$$

Hence $2007^{2^k}+1 \equiv 2 \pmod 4$, and includes exactly 1 power of 2 in its prime factorization. There are 9 such factors.

The other two factors are 2008 and 2006, and we note that

$$\begin{aligned} 2008 &= 2^3 \cdot 251^1, \\ 2006 &= 2^1 \cdot 17^1 \cdot 59^1. \end{aligned}$$

Adding the 4 powers of 2 from these factors to the 9 previous, we see that 2^{13} is the largest power of 2 that divides $2007^{1024}-1$.

11. Represent the number $989 \cdot 1001 \cdot 1007 + 320$ as the product of primes. *Leningrad Mathematical Olympiad*

Introduction to Diophantine Equations

While the name may sound intimidating at first, a **Diophantine equation** is just an equation in which the domains of the variables are confined to the set of integers. Here, we present some problems involving the simplest types of Diophantine Equations – **linear Diophantine equations**, which involve only variables of degree 1.

Some linear Diophantine equations are easy to solve. For instance,

$$x+y=3.$$

We can write the solutions in many ways: $(x, 3-x)$, $(3-y, y)$, $(2007-n, n-2004)$, where each variable represents any integer. Often however, the most useful form of solution uses a new parameter, such as n, with which to write the values of all possible variables. Here is a fairly simple example: $(n, -n+3)$.

1. Find all solutions (x, y) in positive integers to each of the following:
   (a) $2x+3y=40$.
   (b) $2x+3y=4000$.
   (c) $6x+9y=4000$.
   (d) $3x-5y=2008$.
   (e) $13x+14y=2008$.

   The goal in the (standard) general algorithm for solving linear Diophantine equations is to systematically reduce the coefficients of the variables, until the remaining variables can be easily written in parametric form, and the original variables can then be written in terms of those parametric solutions. Notice the relationship between the progression of coefficients and the numbers at each step of the Euclidean Algorithm when finding $\gcd(3,5)$ in the following solution:

Solution to (d): We isolate the variable with the smaller coefficient:

$$3x = 5y + 2008 \qquad \Leftrightarrow \qquad x = y + 669 + \frac{2y+1}{3}.$$

The fraction part is important. Since x and $y + 669$ are both integers, so too is the fraction. We rewrite this fact using a new variable:

$$\frac{2y+1}{3} = a,$$

for some integer a. Then, we have a new, simpler linear Diophantine equation:

$$2y + 1 = 3a.$$

Again, we isolate the variable whose coefficient has a smaller magnitude:

$$2y = 3a - 1 \qquad \Leftrightarrow \qquad y = a + \frac{a-1}{2}.$$

Once again, the fraction part must be an integer, thus

$$\frac{a-1}{2} = b,$$

for some integer n. This yields yet another even more simple Diophantine equation:

$$a - 1 = 2b \qquad \Leftrightarrow \qquad a = 2b + 1.$$

That last step was the turning point. We have the value of a expressed as a parameter of b, where b can be any given integer. Now all we have to do is find x and y in terms of the last parameter b:

$$y = a + \frac{a-1}{2} = (2b+1) + \frac{(2b+1)-1}{2} = 2b + 1 + b = 3b + 1.$$

$$x = y + 669 + \frac{2y+1}{3} = 3b + 700 + \frac{6b+3}{3} = 5b + 701.$$

Hence, for every integer b, there is a solution $(x, y) = (5b + 701, 3b + 1)$.

2. Find all solutions in positive integers for x and y in which both x and y are less than 20 and $5x - 3y = 31$.

3. How many solutions in positive integers for x and y exist such that $3x + 7y = 348$.

4. Using 40 coins, each of which is a quarter, dime, or nickel, find the number of distinct ways to get exactly \$5.00.

5. Each sopper has 3 legs, each junner has 5 legs, and each sennor has 7 legs. In a room there are 30 total soppers, junners, and sennors, with a total of 120 legs. Find all such possible combinations of soppers, junners, and sennors.

1. The number of pairs of positive integers (x, y) which satisfy the equation $x^2 + y^2 = x^3$ is

   (A) 0 (B) 1 (C) 2 (D) not finite (E) none of these

   Answer: (D)

   Solution: Isolating a variable (or more than one) often gives a nice second-look at an algebraic expression. The given equation is equivalent to

   $$y^2 = x^3 - x^2 = x^2(x - 1).$$

   Since x^2 is always a perfect square, any time $x - 1$ is a perfect square, there will be a solution $(x, y) = (x, x\sqrt{x-1})$. Clearly there are infinitely many such solutions.

2. The number of distinct pairs of integers (x, y) such that

   $$0 < x < y \text{ and } \sqrt{1984} = \sqrt{x} + \sqrt{y}$$

   is

   (A) 0 (B) 1 (C) 3 (D) 4 (E) 7

   AHSME

3. Given that $i^2 = -1$, for how many positive integers n is $(n + i)^4$ an integer?

   (A) none (B) 1 (C) 2 (D) 3 (E) 4

   AHSME

4. Suppose p and q are prime numbers such that p divides $q + 1$ and q divides $p + 1$. Determine $p + q$. *Mandelbrot*

5. The Fibonacci number F_{140} is divisible by 13. Find the smallest number $k > 140$ such that F_k is also divisible by 13. *Mandelbrot*

   See if you can generalize the solution to this problem.

6. How many lattice points lie within or on the border of the circle defined in the xy-plane by the equation $x^2 + y^2 = 100$? *iTest*

   Answer: 317

   Solution: We use symmetry to simplify the problem a bit. First, we note that the origin and the 40 points $(\pm n, 0)$ and $(0, \pm n)$ are on or within the circle for integers n such that $1 \leq n \leq 10$. That's 41 lattice points in or on the circle already. Now, we divide the remaining lattice points into regions by quadrant. Since the total number in each quadrant is the same, we just count those in Quadrant I and multiply that total by 4.

Casework helps us as we count the possibilities. We are looking for positive integers x and y such that $x^2 + y^2 \leq 100$, so we consider the cases for x:

$$\begin{aligned}
x = 1 \quad &\Rightarrow \quad y^2 \leq 99 \quad \Rightarrow \quad y \leq 9,\\
x = 2 \quad &\Rightarrow \quad y^2 \leq 96 \quad \Rightarrow \quad y \leq 9,\\
x = 3 \quad &\Rightarrow \quad y^2 \leq 91 \quad \Rightarrow \quad y \leq 9,\\
x = 4 \quad &\Rightarrow \quad y^2 \leq 84 \quad \Rightarrow \quad y \leq 9,\\
x = 5 \quad &\Rightarrow \quad y^2 \leq 75 \quad \Rightarrow \quad y \leq 8,\\
x = 6 \quad &\Rightarrow \quad y^2 \leq 64 \quad \Rightarrow \quad y \leq 8,\\
x = 7 \quad &\Rightarrow \quad y^2 \leq 51 \quad \Rightarrow \quad y \leq 7,\\
x = 8 \quad &\Rightarrow \quad y^2 \leq 36 \quad \Rightarrow \quad y \leq 6,\\
x = 9 \quad &\Rightarrow \quad y^2 \leq 19 \quad \Rightarrow \quad y \leq 4.
\end{aligned}$$

So, there are 9 lattice points $(1, y)$, another 9 of the form $(2, y)$, etc., for a total of

$$9 + 9 + 9 + 9 + 8 + 8 + 7 + 6 + 4 = 69.$$

We multiply this total by 4 to get 276, and add the 41 lattice points from the axes to get $276 + 41 = 317$.

7. Two nonadjacent vertices of a rectangle are $(4, 3)$ and $(-4, -3)$, and the coordinates of the other two vertices are integers. The number of such rectangles is

   (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

   AHSME

8. Consider two solid spherical balls, one centered at $(0, 0, \frac{21}{2})$ with radius 6, and the other centered at $(0, 0, 1)$ with radius $\frac{9}{2}$. How many points (x, y, z) with only integer coordinates (lattice points) are there in the intersection of the balls?

   (A) 7 (B) 9 (C) 11 (D) 13 (E) 15

   AHSME

9. Let S be an integer. Prove that S cannot be expressed as the sum of two or more consecutive positive odd integers if S is prime. *ARML*

10. If the sum $1 + 2 + 3 + \cdots + K$ is a perfect square N^2 and if N is less than 100, find all possible values of K. *AHSME*

11. Three distinct positive Fibonacci numbers, all greater than 1536, are in arithmetic progression. Let N be the smallest possible value of their sum. Find the remainder when N is divided by 2007. *iTest*

    Answer: 501

    Solution: Let F_a, F_b, and F_c be three Fibonacci numbers in arithmetic progression, where $F_a < F_b < F_c$. Note that this means that $a < b < c$ as well. Then $2F_b = F_a + F_c$, so $2F_b > F_c$. However,

    $$2F_b < F_b + F_{b+1} = F_{b+2}.$$

Hence, $F_{b+2} > 2F_b > F_c$, so $b+2 > c$. But $c > b$, so it must be that $c = b+1$. In other words, if three distinct positive Fibonacci numbers are in arithmetic progression, then the two larger of them are consecutive Fibonacci numbers:

$$2F_b = F_a + F_c \qquad \Leftrightarrow \qquad 2F_b = F_a + F_{b+1}.$$

Isolating F_a and simplifying (using the recursive definition of Fibonacci numbers), we get

$$F_a = 2F_b - F_{b+1} = 2F_b - (F_{b-1} + F_b) = F_b - F_{b-1} = F_{b-2}.$$

Thus, the three Fibonacci numbers are F_{b-2}, F_b, and F_{b+1} for any positive integer b.

Now, we want all three of the Fibonacci numbers to be greater than 1536, so we compute some Fibonacci numbers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, \ldots.$$

The arithmetic progression we are looking for is 1597, 4181, 6765, and their sum is

$$1597 + 4181 + 6765 = 3 \cdot 4181 = 12543.$$

So,

$$12543 = 6 \cdot 2007 + 501,$$

where 501 is the remainder.

12. Compute the positive integer value of k that makes the following statement true:

    For all positive integers a, b, and c that make the roots of $ax^2 + bx + c = 0$ rational, the roots of $4ax^2 + 12bx + kc = 0$ will also be rational. *ARML*

13. How many ordered triples of integers (a, b, c) satisfy

    $$|a+b| + c = 19 \;\text{ and }\; ab + |c| = 97?$$

    (A) 0 (B) 4 (C) 6 (D) 10 (E) 12

    AHSME

14. Find the largest positive integer that is equal to the cube of the sum of its digits. *iTest*

    Answer: 19683

    Solution: Let N be the integer we seek, let d be the number of digits of N, and let $S(N)$ be the sum of the digits of N. Since $S(N) \leq 9d$ and $N \geq 10^{d-1}$, we note that

    $$9^3 d^3 \geq [S(N)]^3 = N \geq 10^{d-1}.$$

    From the outer parts of this inequality, $729d^3 \geq 10^{d-1}$, where the right side of the inequality grows much faster than the left as d ranges through the positive integers. In particular, we quickly find that when $d = 7$,

    $$9 \cdot 9 \cdot 9 \cdot 7 \cdot 7 \cdot 7 < 10^6,$$

    and the inequality is violated. A simple induction argument shows us that the inequality is violated for larger values of d as well.

If $d = 6$, then $S(N) \leq 54$, so

$$[S(N)]^3 \leq 54^3 = 157464 \qquad \Rightarrow \qquad N \leq 157464.$$

Note that $S(N) < S(199999) = 46$, thus

$$[S(N)]^3 \leq 45^3 = 91125 \qquad \Rightarrow \qquad N \leq 91125,$$

which contradicts the fact that $d = 6$, so we know that N has at most 5 digits.

Now we search to see if there is at least one value of N with 5 digits that satisfies the desired property. We can place loose bounds on $S(N)$:

$$8000 < N < 125000 \qquad \Leftrightarrow \qquad 8000 < [S(N)]^3 < 125000.$$

Taking the cube roots of the last inequality, we have $20 < [S(N)] < 50$.

We could at this point start cubing potential digit sums, but a little modular arithmetic additionally simplifies our task. More general than the divisibility rule for 9 is the fact that

$$m \equiv S(m) \pmod 9$$

for all positive integers m (a proof of this fact involves writing the value of m according to the place values of its digits). Now, since $N = [S(N)]^3$, we have that

$$N \equiv S(N) \equiv [S(N)]^3 \pmod 9.$$

So,

$$[S(N)]^3 - S(N) \equiv 0 \pmod 9 \qquad \Leftrightarrow \qquad S(N)[S(N)+1][S(N)-1] \equiv 0 \pmod 9.$$

Since $S(N)-1, S(N)$, and $S(N)+1$ are distinct modulo 3, we know that exactly one of them is a multiple of 9. This leaves us with just a few possible values of $S(N)$ that we now cube:

$$\begin{aligned}
26^3 &= 17576 &\rightarrow\quad S(17576) &= 26,\\
27^3 &= 19683 &\rightarrow\quad S(19683) &= 27,\\
28^3 &= 21952 &\rightarrow\quad S(21952) &= 19,\\
35^3 &= 42875 &\rightarrow\quad S(42875) &= 26,\\
36^3 &= 46656 &\rightarrow\quad S(46656) &= 27,\\
37^3 &= 50653 &\rightarrow\quad S(50653) &= 19,\\
44^3 &= 85184 &\rightarrow\quad S(85184) &= 26,\\
45^3 &= 91125 &\rightarrow\quad S(91125) &= 18,\\
46^3 &= 97336 &\rightarrow\quad S(97336) &= 28.
\end{aligned}$$

Thus we see that $27^3 = 19683$ is the largest positive integer that is the cube of the sum of its digits.

15. Find the largest natural number n such that

$$2^n + 2^{11} + 2^8$$

is a perfect square. *iTest TOC*

Answer: 12

Solution: One way to get started is to search first for values of n such that $n \geq 8$. This way, the given quantity is a multiple of 2^8. So, if

$$k^2 = 2^n + 2^{11} + 2^8$$

is a perfect square, then k^2 is a multiple of 2^8. Hence, $16 \mid k$, and so $k = 16m$ for some integer m, and

$$m^2 = 2^{n-8} + 2^3 + 1 = 2^{n-8} + 9.$$

Now we can equate a (factorable) difference of squares and a power of 2:

$$m^2 - 9 = 2^{n-8} \qquad \Leftrightarrow \qquad (m+3)(m-3) = 2^{n-8}.$$

Thus $m+3$ and $m-3$ are powers of 2. Their difference is 6, and 2 is the largest power of 2 that divides 6, so 2 must be $m-3$, hence $m = 5$ is the only possible value of m. Thus,

$$2^{n-8} = 5^2 - 9 = 16 \qquad \Leftrightarrow \qquad n = 12.$$

Since $n = 12$ is the only value of n such that $n \geq 8$ and for which the given expression is a perfect square, 12 is our answer. In fact, $n = 12$ is the only such natural number.

16. Find the sum of all integers n such that

$$n^4 + n^3 + n^2 + n + 1.$$

is a perfect square. *iTest TOC*

Answer: 2

Solution: Looking for ways to approximate the square root of $n^4 + n^3 + n^2 + n + 1$, we note that

$$\begin{aligned}
\left(n^2 + \frac{1}{2}n\right)^2 &= n^4 + n^3 + \frac{1}{4}n^2, \\
\left(n^2 + \frac{1}{2}n + \frac{1}{2}\right)^2 &= n^4 + n^3 + \frac{5}{4}n^2 + \frac{1}{2}n + \frac{1}{4}, \\
\left(n^2 + \frac{1}{2}n + 1\right)^2 &= n^4 + n^3 + \frac{9}{4}n^2 + n + 1.
\end{aligned}$$

We note that

$$(n^4 + n^3 + n^2 + n + 1) - \left(n^4 + n^3 + \frac{1}{4}n^2\right) = \frac{3}{4}n^2 + n + 1 > 0 \tag{1}$$

for all real numbers n, and

$$\left(n^4 + n^3 + \frac{9}{4}n^2 + n + 1\right) - (n^4 + n^3 + n^2 + n + 1) = \frac{5}{4}n^2 \geq 0 \tag{2}$$

for all real numbers n, with equality iff $n = 0$. In summary,

$$\left(n^2 + \frac{1}{2}n\right)^2 < n^4 + n^3 + n^2 + n + 1 \leq \left(n^2 + \frac{1}{2}n + 1\right)^2.$$

So, since either $n^2+\frac{1}{2}n+\frac{1}{2}$ or $n^2+\frac{1}{2}n+1$ is an integer depending on the parity of n, we know that when $n^4+n^3+n^2+n+1$ is a perfect square, it is equal to the square of one of those quantities.

If equality occurs in Inequality 2, then $n=0$, in which case we have $n^4+n^3+n^2+n+1=1^2$. Otherwise, we have that

$$\left(n^2+\frac{1}{2}n+\frac{1}{2}\right)^2=n^4+n^3+\frac{5}{4}n^2+\frac{1}{2}n+\frac{1}{4}=n^4+n^3+n^2+n+1,$$

so

$$\frac{1}{4}n^2-\frac{1}{2}n-\frac{3}{4}=0 \qquad \Leftrightarrow \qquad n^2-2n-3=(n+1)(n-3)=0.$$

Thus, either $n=-1$, in which case $n^4+n^3+n^2+n+1=1^2$, or else $n=3$, in which case $n^4+n^3+n^2+n+1=11^2$. Hence, our answer is $0+(-1)+3=2$.

The Euclidean Algorithm Revisited

1. Find all positive integers p such that
$$\frac{3p+25}{2p-5}$$
is also a positive integer. *AHSME*

2. Find the least positive integer n for which $\frac{n-13}{5n+6}$ is a non-zero reducible fraction.

   (A) 45 (B) 68 (C) 155 (D) 226 (E) none of these

   AHSME

   Answer: (E)

   Solution: We can restate the problem as follows: Find the least positive integer n such that $n-13$ and $5n+6$ have a GCD greater than 1. Now, as a GCD problem, we apply the Euclidean Algorith until one of the two algebraic quantities becomes a constant:

$$(5n+6)-5(n-13)=71 \qquad \Leftrightarrow \qquad \gcd(5n+6,n-13)=\gcd(71,n-13).$$

   Now we want $\gcd(71,n-13)>1$, and since 71 is prime, the GCD can be only 1 or 71. So, our task is simply to find the smallest positive integer n such that $n-13$ is a multiple of 71, and the given fraction is non-zero. So, $n-13=1\cdot 71 \Rightarrow n=84$ is the answer.

3. Prove that any pair of consecutive Fibonacci numbers is relatively prime.

Number Sense

Many people think of *number sense* as a series of tricks with little practical or lasting value. Another view of number sense is that a more complete understanding of mathematics makes many problems simple – problems that less diligent students find difficult or impossible. What do you think?

1. Find the value of $\sqrt{72 \cdot 128 \cdot 75 \cdot 108}$.
2. Find the total number of digits in the product $8^{13} \cdot 25^{22}$.
3. What is the least natural number, greater than 1, that is a factor of $11000 + 1100 + 11$? *MATHCOUNTS*
4. Find the GCD of 642 and 32172.
5. Compute the value of $50^2 - (100)(57) + 57^2$.
6. The six-digit decimal integer $349AB1$ is a perfect square, where A and B are decimal digits, not necessarily distinct. Find its three-digit square root.
7. When 5137247 is multiplied by 143215, the result is $72573A829214$. Find the digit A.

Divisibility – More Advanced Concepts

1. A set of consecutive positive integers beginning with 1 is written on a blackboard. One number is erased. The average (arithmetic mean) of the remaining numbers is $35\frac{7}{17}$. What number was erased? *AHSME*
2. The product of any two of the positive integers 30, 72, and N is divisible by the third. What is the smallest possible value of N? *Mandelbrot*
3. Compute the smallest positive integer $n > 100$ such that $\binom{n}{101}$ is divisible by $\binom{n}{100}$, but is not equal to it. *ARML*
4. What is the smallest positive integer k such that the number $\binom{2k}{k}$ ends in two zeros? *iTest*

   Answer: 13

   Solution: We want 100 to divide $\binom{2k}{k}$. Let's count the number of multiples of 5 in the numerator and the denominator. The numerator has $\lfloor \frac{2k}{5} \rfloor + \lfloor \frac{2k}{25} \rfloor + \cdots$ and the denominator has $2\lfloor \frac{k}{5} \rfloor + 2\lfloor \frac{k}{25} \rfloor + \cdots$, where $\lfloor x \rfloor$ represents the largest integer smaller than x (i.e. we round down). We want the first expression to be larger than the second by 2. Clearly $\lfloor \frac{2k}{5^n} \rfloor$ is no more than one larger than $2\lfloor \frac{k}{5^n} \rfloor$, so to get 2 we need $2k$ to be divisible by 25 one more time than k is. The first time that happens is for $k = 13$. Using the same idea, we easily check that the power of 2 dividing the numerator is more than two larger than the power of 2 dividing the denominator, so the answer is 13.

5. Let N be the smallest positive integer such that $2008N$ is a perfect square and $2007N$ is a perfect cube. Find the remainder when N is divided by 25.

   Answer: 17

   Solution: We construct N using what we know about prime factorizations of the given square and cube. We note that

$$\begin{aligned} 2008N &= 2^3 \cdot 251^1 \cdot N, \\ 2007N &= 3^2 \cdot 223^1 \cdot N, \end{aligned}$$

where 2, 3, 223, and 251 are all prime.

Since $2008N$ is a perfect square, the exponents in its prime factorization are all even, so

$$N = 2^{2a_1+1} \cdot 3^{2a_2} \cdot 223^{2a_3} \cdot 251^{2a_4+1} \cdot x^2,$$

where x is either 1 or composed entirely of primes other than the four we used. Since $2007N$ is a perfect square, the exponents in its prime factorization are all multiples of 3, so

$$N = 2^{3b_1} \cdot 3^{3b_2+1} \cdot 223^{3b_3+2} \cdot 251^{3b_4} \cdot y^3,$$

where y is either 1 or composed entirely of primes other than the four we used.

Now, we find the smallest possible exponents. The two prime factorizations of N above give us systems of linear congruences to solve, though the least possible exponents are small enough that we can just hunt for them. Also, the smallest N occurs when $x = y = 1$, so

$$N = 2^3 \cdot 3^4 \cdot 223^2 \cdot 251^3.$$

Finally, we find the remainder when N is divided by 25:

$$\begin{aligned} N &= 2^3 \cdot 3^4 \cdot 223^2 \cdot 251^3 \\ &\equiv 2^3 \cdot 3^4 \cdot (-2)^2 \cdot 1^3 \\ &\equiv 8 \cdot 81 \cdot 4 \cdot 1 \\ &\equiv 8 \cdot 6 \cdot 4 \equiv 192 \equiv 17 \pmod{25}. \end{aligned}$$

So, 17 is the remainder.

6. For each positive integer n, let $S_n = \sum_{k=1}^{n} k^3$, and let $d(n)$ be the number of positive divisors of n. For how many positive integers m, where $m \leq 25$, is there a solution n to the equation $d(S_n) = m$? *iTest TOC*

Answer: 6

Solution: First, we can find a closed form expression for S_n:

$$S_n = \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right].$$

Now, we consider what this tells us about the possible values of $d(S_n)$. Since S_n is a perfect square, the value of m must be odd. We also note for $n > 2$, that S_n is the square of a composite number, because

$$\frac{n(n+1)}{2} = \frac{n}{2} \cdot (n+1) = n \cdot \frac{n+1}{2},$$

where one of the two products is the product of two integers. Since $\gcd(n, n+1) = 1$, these integers are in fact relatively prime, so the prime factorization of S_n includes at least two primes for $n > 2$. So, when we compute $d(S_n)$, adding 1 to each of the exponents of the prime factorization of S_n, then taking the product of the results, we wind up with a composite number (an odd composite number in particular).

Now we take care of the special cases where $n \leq 2$. We note that $d(S_1) = d(1) = 1$ and that $d(S_2) = d(9) = 3$. So, there are solutions to the given equation when $m = 1$ or $m = 3$. Next, we consider which odd composites can be values of m and yield solutions. Just checking a few values of n, we find that

$$\begin{aligned} d(S_3) = d(6^2) &= d(2^2 \cdot 3^2) = (2+1)(2+1) = 9, \\ d(S_7) = d(28^2) &= d(2^4 \cdot 7^2) = (4+1)(2+1) = 15, \\ d(S_8) = d(36^2) &= d(2^4 \cdot 3^4) = (4+1)(4+1) = 25. \end{aligned}$$

Many solvers probably used more creative means to hunt down values of n to match the possible odd composite values of m.

The only possible odd composite left in the range of m is $m = 21$. We need to find an n such that $d(S_n) = 21$, or show that no such value exists. When $d(S_n) = 21$, we know that

$$S_n = (p^3 \cdot q^1)^2,$$

for some primes p and q. Looking to construct an example, we note that $2^4 + 1$ is prime, so

$$S_{16} = \left(2^3 \cdot 17^1\right)^2 \qquad \Rightarrow \qquad d(S_{16}) = 21.$$

Finally, we note that there are 6 possible values of m (1, 3, 9, 15, 21, 25) where $m \leq 25$ and there is a solution to the equation $d(S_n) = m$.