This document is a sample of the MIST Academy Intermediate Problem Solving curriculum. Your child/student may find some of these problems easy and others very difficult. Students are working with the right curriculum when they get some problems and struggle with others.

Algebra

MIST Academy curriculum covers those topics in a standard Algebra II course that are most difficult to students, and introduces many new topics. The art of algebraic manipulation is a focus of this course.

1. Suppose that b and c are constants and
$$(x+2)(x+b) = x^2 + cx + 6.$$
What is c?
(A) -5 (B) -3 (C) -1 (D) 3 (E) 5
(AHSME)

2. A parabola $y = ax^2 + bx + c$ has vertex $(4, 2)$, and $(2, 0)$ is on the graph of the parabola. What is abc?
(A) -12 (B) -6 (C) 0 (D) 6 (E) 12
(AHSME)

3. The sum of two numbers is 10, and their product is 21. Find the sum of their squares.

4. Find the remainder that results when
$$(x+1)^5 + (x+2)^4 + (x+3)^3 + (x+4)^2 + (x+5)$$
is divided by $x + 2$. *(ARML)*

5. The product of three consecutive positive integers is 8 times their sum. What is the sum of their squares?
(A) 50 (B) 77 (C) 110 (D) 149 (E) 194
(AMC 10)

6. If each boy purchases a patty and each girl purchases a bun, they would spend a total of one cent more than if each boy purchased a bun and each girl purchased a patty. We know that the number of boys is greater than the number of girls. What is the difference between the number of boys and the number of girls? *(Leningrad Mathematical Olympiad)*

7. How many integers x satisfy the inequality
$$(x-2006)(x-2004)(x-2002)\cdots(x-4) < 0?$$
(Alabama ARML TST)

8. Find all real values of x which satisfy

$$\frac{1}{x+1} + \frac{6}{x+5} \geq 1.$$

(ARML)

9. Suppose that m and n are positive integers such that $m < n$, the geometric mean of m and n is greater than 2007, and the arithmetic mean of m and n is less than 2007. How many pairs (m, n) satisfy these conditions? *(iTest)*

10. Compute $\displaystyle\sum_{k=1}^{5}\sum_{n=1}^{6} kn$. *(Alabama ARML TST)*

11. Let $\lfloor a \rfloor$ be the greatest integer less than or equal to a and let $\{a\} = a - \lfloor a \rfloor$. Find $10(x+y+z)$ given that

$$\begin{aligned} x + \lfloor y \rfloor + \{z\} &= 14.2, \\ \lfloor x \rfloor + \{y\} + z &= 15.3, \\ \{x\} + y + \lfloor z \rfloor &= 16.1. \end{aligned}$$

(Alabama ARML TST)

12. Suppose that $x^3 + px^2 + qx + r$ is a cubic with a double root at a and another root at b, where a and b are real numbers. If $p = -6$ and $q = 9$, what is r? *(iTest)*

13. Find the value of c such that the system of equations,

$$\begin{aligned} |x + y| &= 2007, \\ |x - y| &= c, \end{aligned}$$

has exactly two solutions (x, y) in real numbers. *(iTest)*

Counting

Counting techniques includes casework, combinations, complementary counting, the "greedy algorithm", permutations, the principle of inclusion-exclusion, and the use of Venn diagrams. Students study these topics a little more in-depth in the Intermediate class.

1. A restaurant offers three desserts, and exactly twice as many appetizers as main courses. A dinner consists of an appetizer, a main course, and a dessert. What is the least number of main courses that the restaurant should offer so that a customer could have a different dinner each night for a year? *(AMC 10)*

2. A large cube is painted green and then chopped up into 64 smaller congruent cubes. How many of the smaller cubes have at least one face painted green? *(Vestavia ARML TST)*

3. A 2×2 square grid is constructed with four 1×1 squares. The square on the upper left is labeled A, the square on the upper right is labeled B, the square on the lower left is labled C, and the square on the lower right is labeled D. The four squares are to be painted such that 2 are blue, 1 is red, and 1 is green. In how many ways can this be done? *(Alabama ARML TST)*

4. Find the number of six-digit positive integers for which the digits are in increasing order. *(Alabama ARML TST)*

5. The increasing sequence $2, 3, 5, 6, 7, 10, 11, \ldots$ consists of all positive integers that are neither the square nor the cube of a positive integer. Find the 500^{th} term of this sequence. *(AIME)*

6. Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

7. The number 915 is a 3-digit number the sum of whose digits is 15. Using only the non-zero digits, how many different positive 3-digit base 10 numerals (none of which uses the same digit more than once in its 3-digit base 10 representation) can be formed which have a digital sum of 15? *(ARML)*

8. One commercially available ten-button lock may be opened by depressing – in any order – the correct five buttons. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow? *(AIME)*

9. A spider has one sock and one shoe for each of its eight legs. On each leg the sock must be put on before the shoe. In how many different orders can the spider put on its socks and shoes? *(AMC 10/12)*

Complex Numbers

In the Intermediate Problem Solving class, students study the arithmetic, algebra, and geometry of complex numbers.

1. Define i such that $i^2 = -1$. What is $(i - i^{-1})^{-1}$? *(AHSME)*

2. Find the distance between $3 + 7i$ and $-6 + 47i$ in the complex plane.

3. Find the solutions to $z^2 + 3z = 9 + 7i$.

4. The six solutions of $x^6 = -64$ are written in the form $a + bi$, where a and b are real numbers. What is the product of those solutions with $a > 0$. *(AHSME)*

5. Two solutions of $x^4 - 3x^3 + 5x^2 - 27x - 36 = 0$ are pure imaginary numbers. Find these two solutions. *(ARML)*

6. Find all complex numbers z such that $5z + 12\overline{z} = 8 - 15i$.

7. Evaluate
$$i + 2i^2 + 3i^3 + \cdots + 2007i^{2007}.$$

8. Evaluate $(i + 1)^{1000} - (i - 1)^{1000}$.

9. Given that $w = 5 + 3i$ and $z = 8 + 11i$. Find the area of the convex quadrilateral formed by w, $\overline{w}$, z, and $\overline{z}$ in the complex plane.

10. Find the area bounded by the locus of points z in the complex plane such that $|z - 7i| = 4$.

11. Suppose that n is a positive integer and that $z + \frac{1}{z} = 2\cos\theta$, where $0 < \theta < \pi$. What is the value of $z^n + \frac{1}{z^n}$? *(AHSME)*

12. Find c if a, b, and c are positive integers which satisfy $c = (a + bi)^3 - 107i$, where $i^2 = -1$. *(AIME)*

13. Show that there is exactly one complex number z such that
$$|z - 3| = |z + 2i| = |z + 1 - 4i|.$$

14. Let $\omega^3 = 1$, where $\omega \neq 1$, find the value of
$$(1 - \omega + \omega^2)(1 + \omega - \omega^2).$$

Games and Puzzles

While many educators are skilled at the use of games and puzzles in the classroom, games and puzzles may be the single most under-appreciated area of pre-college mathematics. Games and puzzles excite both curious and competitive minds and often lead talented students to a deeper understanding of mathematics. The following are just a few of the games and puzzles that will be explored in the MIST Academy classroom.

1. Aliquot part is another name for a proper divisor, i.e. any divisor of a given number other than the number itself. A prime number has only one aliquot part – the number 1. 1, 2, 3, 4, 6 are all aliquot parts of 12. The number 1 does not have aliquot parts. In the Aliquot game, players take turns subtracting an aliquot part of the number left by their opponent. The winner is the last player able to perform such a subtraction. The loser is the player left with a number that has no aliquot parts – 1. Thus the objective of the game is to leave your opponent without a move.

2. Two players play the following game on a 9×9 chessboard. They write in succession one of two signs in any free square of the board; the player making the first move writes a plus sign and the other player writes a minus sign. What all the squares of the board are filled, the scores of the players are evaluated. The number of rows and columns containing more plus signs than minus signs is the score of the first player, and the number of all other rows and columns is the score of the second player. What si the highest number of points the first player can gain in an errorless game? *(Leningrad Mathematical Olympiad)*

3. Amy and Martin play a game in which they take turn flipping a fair coin until one of them flips a head. The first person to flip a head is the winner. If Amy goes first, find the probability that she wins.

4. A pawn is placed in the central square of a 11×11 chessboard. Two players move the pawn in succession to any other square, but each move (beginning with the second) must be longer than the previous one. The player who cannot make such a move loses. Who wins in an errorless game? *(Leningrad Mathematical Olympiad)*

Geometry

The Intermediate Problem Solving curriculum includes and extends those topics taught in an advanced geometry classroom.

1. The difference between the areas of the circumcircle and incircle of an equilateral triangle is 300π square units. Find the number of units in the length of a side of the triangle.

2. An equilateral triangle with side length 1 has the same area as a square with side length s. Find s. *(iTest)*

3. The median of a trapezoid cuts the trapezoid into two regions whose areas are in the ratio $1:2$. Compute the ratio of the smaller base of the trapezoid to its longer base. *(ARML)*

4. The 3 altitudes of acute scalene triangle ABC intersect at P. If $AB = x$, $CP = y$, and d represents the length of the diameter of the circle circumscribed about triangle ABC, write an EQUATION expressing d explicitly in terms of x and y. *(ARML)*

5. In triangle ABC, the perpendicular bisector of $\overline{AC}$ intersects $\overline{AC}$ at M and $\overline{AB}$ at T. If the area of triangle AMT is $1/4$ the area of triangle ABC, and $\angle A + \angle C = 128^\circ$, compute the number of degrees in angle A. *(ARML)*

6. A square is inscribed in a circle of diameter 12. Two vertices of a triangle are also vertices of one side of the square. The other vertex of the triangle is on the circle. Find the largest possible area of the triangle. *(Alabama ARML TST)*

7. Form a pentagon by taking a square of side length 1 and an equilateral triangle of side length 1 and placing the triangle so that one of its sides coincides with a side of the square. Then "circumscribe" a circle around the pentagon, passing through three of its vertices, so that the circle passes through exactly one vertex of the equilateral triangle, and exactly two vertices of the square. What is the radius of the circle? *(iTest)*

8. Find the area of an equiangular octagon, the lengths of whose sides are alternately 1 and $\sqrt{2}$. *(ARML)*

9. Four equilateral triangles are drawn such that each one shares a different side with a square of side length 10. None of the areas of the triangles overlap with the area of the square. The four vertices of the triangles that aren't vertices of the square are connected to form a larger square. Find the area of this larger square. *(Alabama ARML TST)*

10. In concave hexagon $ABCDEF$, $\angle A = \angle B = \angle C = 90^\circ$, $\angle D = 100^\circ$, and $\angle F = 80^\circ$. Also, $CD = FA$, $AB = 7$, $BC = 10$, and $EF + DE = 12$. Compute the area of the hexagon. *(Alabama ARML TST)*

Number Sense

Many teachers erroneously dismiss number sense as a collection of worthless tricks. Poorly understood and taught, this might be the case. At MIST Academy we help students see the arithmetic, algebra, and number theory behind methods that make nearly all of computation, math, and problem solving easier.

1. Compute $65 \cdot 123 - 25 \cdot 123$.

2. Find the value of $\sqrt{72 \cdot 128 \cdot 75 \cdot 108}$.

3. Compute $2007^2 - 1993^2$.

4. Find the total number of digits in the product $8^{13} \cdot 25^{22}$.

5. Compute the value of A divided by B given that $A = 666666^6$ and $B = 222222^6$.

6. Find the GCD of 642 and 32172.

7. Compute $\dfrac{3^{2008} - 3^{2005}}{3^{2005} - 3^{2002}}$.

8. Compute the value of $50^2 - (100)(57) + 57^2$.

9. Compute $91^3 - 3 \cdot 91^2 + 3 \cdot 91 - 1$.

10. The six-digit decimal integer $349AB1$ is a perfect square, where A and B are decimal digits, not necessarily distinct. Find its three-digit square root.

11. Let $N = \underline{9}\,\underline{9}\,\underline{9}\,\underline{9} \cdots \underline{9}\,\underline{9}\,\underline{9}$, where the digit 9 occurs 220 times. Compute the *sum* of the digits of the number N^2. *(ARML)*

12. If a, b, c, and n are positive integers with $a < 11$, and $n^a + n^b - n^c = 0$, compute the maximum possible value for $a^n + b^n - c^n$. *(ARML)*

Number Theory

Number theory, the mathematics of integers is one of the most unfortunately under-covered areas of mathematics, especially at the middle school level. In a world in which computing is largely based on logic and number theory, MIST Academy promotes number theory as a substantial part of its curriculum.

1. Find the units digit of the sum
$$(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2 + \cdots + (2007!)^2.$$
(iTest)

2. For how many positive integers n is $n^2 - 3n + 2$ a prime number?
(A) none (B) one (C) two
(D) more than two, but finitely many (E) infinitely many
(AMC 10)

3. Find the value of $x + y$ where x and y are positive integers that satisfy
$$13x + 19y = 212.$$

4. Find $a + b$ given that the five-digit integer $3ab16$ is a perfect square.

5. For how many ordered pairs of digits (A, B) is $2AB8$ a multiple of 12? *(Alabama ARML TST)*

6. How many of the positive divisors of 3,240,000 are perfect cubes? *(Alabama ARML TST)*

7. Let a/b be the probability that a randomly selected divisor of 2007 is a multiple of 3. If a and b are relatively prime positive integers, find $a + b$. *(iTest)*

8. Find the number of ordered pairs of integers (x, y) which satisfy
$$x^2 + 4x + y^2 = 21.$$
(Alabama ARML TST)

9. Let N be the smallest positive integer such that $2008N$ is a perfect square and $2007N$ is a perfect cube. Find the remainder when N is divided by 25. *(iTest)*

10. Consider the "tower of power" $2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$, where there are 2007 twos including the base. What is the last (units) digit of this number? *(iTest)*

11. Find the largest integer n such that $2007^{1024} - 1$ is divisible by 2^n. *(iTest)*

Probability

1. Jenny pulls two marbles out of a bag containing 4 red and 6 green marbles. Find the probability that the two marbles are of the same color.

2. If a, b and c are three (not necessarily different) numbers chosen randomly and with replacement from the set $\{1, 2, 3, 4, 5\}$, the probability that $ab + c$ is even is

   $$\text{(A) } \frac{2}{5} \qquad \text{(B) } \frac{59}{125} \qquad \text{(C) } \frac{1}{2} \qquad \text{(D) } \frac{64}{125} \qquad \text{(E) } \frac{3}{5}$$

   (AHSME)

3. A fair six-sided die is rolled four times. Find the probability that at least two of the results are the same.

4. My frisbee group often calls "best of five" to finish our games when it's getting dark, since we don't keep score. The game ends after one of the two teams scores three points (total, not necessarily consecutive). If every possible sequence of scores is equally likely, what is the expected score of the losing team? *(iTest)*

5. Two six-sided dice are constructed such that each face is equally likely to show up when rolled. The numbers on the faces of one of the dice are 1, 3, 4, 5, 6, and 8. The numbers on the faces of the other die are 1, 2, 2, 3, 3, and 4. Find the probability of rolling a sum of 9 with these two dice. *(Alabama ARML TST)*

6. Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red, and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same? *(AMC 10)*

7. River draws four cards from a standard 52 card deck of playing cards. Exactly 3 of them are 2's. Find the probability River drew exactly one spade and one club from the deck. *(Alabama ARML TST)*

8. James writes down fifteen 1's in a row and randomly writes + or − between each pair of consecutive 1's. One such example is

   $$1 + 1 + 1 - 1 - 1 + 1 - 1 + 1 - 1 - 1 - 1 - 1 + 1 + 1 - 1.$$

   What is the probability that the value of the expression James wrote down is 7? *(iTest)*

9. When a certain biased coin is flipped five times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. Let $\frac{i}{j}$, in lowest terms, be the probability that the coin comes up heads in exactly 3 out of 5 flips. Find $i + j$. *(AIME)*

Sequences & Series

Problems involving sequences and series can challenge a student to apply master of arithmetic, pattern hunting, algebra, number theory, and sometimes other areas of mathematics. These problems often frustrate even talented solvers, but can be the most enjoyable at the same time.

1. Evaluate
$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots.$$

2. Find all integers k such that $k+(k+2)+(k+4)+\cdots+50+52=462$. (32 and -30.)

3. Compute the sum
$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots+\frac{1}{9900}.$$

4. Let (a_k) be a sequence of integers such that $a_1=1$ and $a_{m+n}=a_m+a_n+mn$ for all positive integers m and n. Find a_{12}. *AMC 10*

5. Is it possible to write the natural numbers $1,2,\ldots,100$ in a row so that the difference between any two adjacent numbers is no less than 50? *(Leningrad Mathematical Olympiad)*

6. Suppose x,y,z is a geometric sequence with common ratio r and $x\neq y$. If $x,2y,3z$ is an arithmetic sequence, find the value of r. *AHSME*

7. A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term in the geometric progression? *AMC 12*

8. Let a_1, a_2, $\ldots$, and b_1, b_2, $\ldots$ be arithmetic progressions such that $a_1=25$, $b_1=75$, and $a_{100}+b_{100}=100$. Find the sum of the first 100 terms of the progression a_1+b_1, a_2+b_2, $\ldots$. *AHSME*

9. Find the value of $a_2+a_4+a_6+\cdots+a_{98}$ if a_1, a_2, a_3, $\ldots$ is an arithmetic progression with common difference 1, and $a_1+a_2+a_3+\cdots+a_{98}=137$. *AIME*

10. The sequence a_1, a_2, a_3, $\ldots$ satisfies $a_1=19$, $a_9=99$, and, for all $n\geq 3$, a_n is the arithmetic mean of the first $n-1$ terms. Find a_2. *AHSME*

11. We draw a broken line figure as follows:

    Start at the origin and –
    go 1 unit in the 60° direction; then
    go 1/2 unit in the $-60°$ direction; then

go 1/4 unit in the 60° direction; then
go 1/8 unit in the −60° direction; and so on.
Each segment is 1/2 the length of the previous one, and each direction is an angle measured relative to the positive x-axis (with directions alternating as noted).

The broken line approaches a unique "limiting point" whose coordinates are $x = a$ and $y = b$. Compute the ordered pair (a, b).

12. Consider the triangular array of numbers 0, 1, 2, 3, ... along the sides and interior numbers obtained by adding the two adjacent numbers in the previous row. Rows 1 through 6 are shown.

					0					
				1		1				
			2		2		2			
		3		4		4		3		
	4		7		8		7		4	
5		11		15		15		11		5

Let $f(n)$ denote the sum of the numbers in row n. What is the remainder when $f(100)$ is divided by 100? *(AHSME)*

13. If P is the product of n quantities in geometric progression, S their sum, and S' the sum of their reciprocals, then find P in terms of S, S', and n. *(AHSME)*

14. One day Jason finishes his math homework early, and decides to take a jog through his neighborhood. While jogging, Jason trips over a leprechaun. After dusting himself off and apologizing to the odd little magical creature, Jason, thinking there is nothing unusual about the situation, starts jogging again. Immediately the leprechaun calls out, "hey, stupid, this is your only chance to win gold from a leprechaun!"

    Jason, while not particularly greedy, recognizes the value of gold. Thinking about his limited college savings, Jason approaches the leprechaun and asks about the opportunity. The leprechaun hands Jason a fair coin and tells him to flip it as many times as it takes to flip a head. For each tail Jason flips, the leprechaun promises one gold coin.

    If Jason flips a head right away, he wins nothing. If he first flips a tail, then a head, he wins one gold coin. If he's lucky and flips ten tails before the first head, he wins *ten gold coins*. What is the expected number of gold coins Jason wins at this game? *(iTest)*