The 1st Annual iTest Tournament of Champions

Round 4 Problems

1. (3 points) A fair 20-sided die has faces numbered 1 through 20. The die is rolled three times and the outcomes are recorded. If a and b are relatively prime integers such that a/b is the probability that the three recorded outcomes can be the sides of a triangle with positive area, find $a+b$.

2. (4 points) Let m be the maximum possible value of $x^{16}+\dfrac{1}{x^{16}}$, where

$$x^6 - 4x^4 - 6x^3 - 4x^2 + 1 = 0.$$

Find the remainder when m is divided by 2007.

3. (5 points) A sequence $a_1, a_2, a_3, \ldots$ is defined as follows: $a_1 = 2007$, and $a_n = a_{n-1} + n \pmod{k}$, where $0 \le a_n < k$. For how many values of k, where $2007 < k < 10^{12}$ does the sequence assume all k possible values (modulo k residues)?

4. (6 points) Bobby Fisherman played a tournament in which he played 2009 players. He either won or lost every game. He lost his first two games, but won 2002 total games. At the conclusion of each game, he computed his exact winning percentage at that moment. Let $w_1, w_2, \ldots, w_{2009}$ be his winning percentages after games 1, 2, ..., 2009 respectively. There are some real numbers, such as 0, which are necessarily members of the set $W = \{w_1, w_2, \ldots, w_{2009}\}$. How many positive real numbers are necessarily elements of set W, regardless of the order in which he won or lost his games?

5. (7 points) Convex quadrilateral $ABCD$ has the property that the circles with diameters AB and CD are tangent at point X inside the quadrilateral, and likewise, the circles with diameters BC and DA are tangent at a point Y inside the quadrilateral. Given that the perimeter of $ABCD$ is 96, and the maximum possible length of XY is m, find $\lfloor 2007m \rfloor$.