Alabama ARML Team Selection Test

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The goal of this solutions package is not so much to present a proof of each answer, but to try to motivate the methods and techniques used in solving them. For this reason, the solutions are longer than is necessary, but hopefully more beneficial to students.

1. Students with a knowledge of generating functions might recognize that the dice have the same exact distribution of sums as an ordinary pair of dice. However, we can solve the problem via casework. The ways in which we can get a sum of 9 are (first die + second die)

$$\begin{aligned} 5+4 &= 9 \\ 6+3 &= 9 \\ 6+3 &= 9 \\ 8+1 &= 9 \end{aligned}$$

   We list $6+3$ twice because there are two faces labeled 3 on the second die. There are $6 \cdot 6 = 36$ possible rolls (where we consider all faces distinct outcomes), so the probability is $4/36 = \boxed{1/9}$.

2. Complementary counting is the easiest method for this problem. The cubes with no painted faces make up a $2 \times 2 \times 2$ cube within the $4 \times 4 \times 4$ cube. There are 2^3 cubes without a painted face, so the total number of cubes with at least one face painted is

$$4^3 - 2^3 = 64 - 8 = \boxed{56}.$$

3. Through rotational symmetry, we can see that the centers of the circles are the same. Let the center of the circles be O, and label two of the vertices of the triangle A and B. Let M be the midpoint of $\overline{AB}$. Note the $\overline{AO}$ bisects $\angle A$, so AOM is a 30-60-90 triangle. This tells us that the inradius $r = OM$ is half the length of the circumradius $R = OA$. Now we have an equation we can solve:

$$\pi R^2 - \pi r^2 = 4\pi r^2 - \pi r^2 = 3\pi r^2 = 300\pi \quad \Rightarrow \quad r = 10.$$

   Again from the 30-60-90 triangle we know that $AM/OM = \sqrt{3}$. Since $OM = r = 10$, $AM = 10\sqrt{3}$. Since M is the midpoint of AB, we have $AB = 20\sqrt{3}$ as the length of a side of the triangle.

4. We are given that $2AB8 \equiv 0 \pmod{12}$, and since $12 = 3 \cdot 4$ where 3 and 4 are relatively prime, we have

$$\begin{aligned} 2AB8 &\equiv 0 \pmod{3} \\ 2AB8 &\equiv 0 \pmod{4} \end{aligned}$$

Using the divisibility rules for 3 and 4, this congruences simplify to

$$\begin{aligned} A+B+10 &\equiv A+B+1 \equiv 0 \pmod{3} \\ B8 &\equiv 10B+8 \equiv 2B \equiv 0 \pmod{4} \end{aligned}$$

Now we perform casework. From the second congruence, we know that B is even. From the first, we know that $A+B \equiv 2 \pmod{3}$ which helps us tally the cases for B:

$$\begin{aligned} B=0 &\quad\rightarrow\quad (2,0),(5,0),(8,0) \\ B=2 &\quad\rightarrow\quad (0,2),(3,2),(6,2),(9,2) \\ B=4 &\quad\rightarrow\quad (1,4),(4,4),(7,4) \\ B=6 &\quad\rightarrow\quad (2,6),(5,6),(8,6) \\ B=8 &\quad\rightarrow\quad (0,8),(3,8),(6,8),(9,8) \end{aligned}$$

There are a total of $3+4+3+3+4 = \boxed{17}$ solutions.

5. The labeling of the squares A, B, C, D just means that each pattern is equivalent to an ordering of the colors. We are looking for ways to arrange bbrg (the first letter of each color). The number of arrangements of these letters is

$$\frac{4!}{2!1!1!} = \frac{24}{2} = \boxed{12}.$$

6. Prime factorization helps us organize information about the divisors of integers.

$$3240000 = 2^7 \cdot 3^4 \cdot 5^4.$$

Any divisor of 3240000 that is a perfect cube must have a prime factorization in the form

$$2^a \cdot 3^b \cdot 5^c,$$

where $0 \le a \le 7$, $0 \le b \le 4$, and $0 \le c \le 4$. However, each of the exponents much be a multiple of 3 since we are looking for perfect cubes. There are 3 possible values for a (0, 3, 6), 2 for each b and c (0 or 3), so the total number of divisors that are perfect cubes is

$$3 \cdot 2 \cdot 2 = \boxed{12}.$$

7. Before getting into the solution to this one, let's take a look at a derivation that can be generalized to all geometric series. Take the series sum,

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots .$$

The key to summing this series is to compare consecutive terms in a meaningful way. We do this by multiplying the entire series by the common ratio:

$$\frac{S}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots .$$

Subtracting this new series from the previous allows a telescoping of terms and the result is $S/2 = 1/2$, so $S = 1$ is the sum of the series.

A similar process can be applied to the more generalized problem of summing series with polynomial numerators. For instance,

$$S = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \cdots .$$

Multiplying this series by 1/2 we have

$$\frac{S}{2} = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots + \frac{n}{2^{n-1}} + \cdots .$$

Subtracting the second series from the first does not telescope the series immediately, but does decrease the degree of the polynomial in the numerator of the general series:

$$\frac{S}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots .$$

Now we can sum the geometric series to find that $S = 2$ is the sum of the original series.

This method of multiplying by the common ratio and subtracting will always decrease the degree of the general polynomial numerator by 1, allowing us to continually simplify a series sum to something common – a geometric series. Students interested in learning exactly why might read up on finite differences as they relate to polynomials.

Now, let's apply this method to the problem at hand:

$$S = \frac{6}{2} + \frac{11}{4} + \frac{18}{8} + \frac{27}{16} + \cdots + \frac{n^2 + 2n + 3}{2^n} + \cdots .$$

We could break the general term up into individual pure polynomial parts, but let's see how the subtraction process works with everything kept together. Multiplying by 1/2 we have

$$\frac{S}{2} = \frac{6}{4} + \frac{11}{8} + \frac{18}{16} + \cdots .$$

Subtracting the second equation from the first we have

$$\frac{S}{2} = \frac{6}{2} + \frac{5}{4} + \frac{7}{8} + \frac{9}{16} + \cdots .$$

Not breaking the pieces of the polynomial up into a sum of pure polynomials means that the pattern of the general term of the numerator emerges only after the first term, but we do see that the pattern is linear (odd integers). Now, we multiply this new series by 1/2:

$$\frac{S}{4} = \frac{6}{4} + \frac{5}{8} + \frac{7}{16} + \frac{9}{32} + \cdots .$$

Subtracting this newest series from the previous gives

$$\frac{S}{4} = \frac{6}{2} - \frac{1}{4} + \frac{2}{8} + \frac{2}{16} + \frac{2}{32} + \cdots .$$

The general pattern emerges after the first two terms and is an ordinary geometric series with sum 1/2, so

$$\frac{S}{4} = \frac{6}{2} - \frac{1}{4} + \frac{1}{2} = \frac{13}{4} \quad \Rightarrow \quad S = \boxed{13}.$$

8. We have squared terms, so we begin by completing the square for the x terms. We do this by adding 4 to both sides of the equation:

$$(x+2)^2 + y^2 = 25$$

Now, let $z = x+2$. This simplifies the problem to finding ordered pairs of integers (z, y) such that

$$z^2 + y^2 = 25.$$

Since the square of an integer must be at least 0 (by the trivial inequality), we have only a few cases to check. We find that the squares are either 0 and 25 or 9 and 16. This leads to 12 solutions for (z, y):

$$(\pm 5, 0), (0, \pm 5), (\pm 3, \pm 4), (\pm 4, \pm 3),$$

so there are $2 + 2 + 4 + 4 = \boxed{12}$ total solutions.

9. There are a number of ways to work this problem, but laying the diagram out on a coordinate plane helps make the explanation clear. Center the circle at the origin so that it has the equation $x^2 + y^2 = 36$. Arrange the square so that its sides lie on the lines $x = \pm 3\sqrt{2}$ and $y = \pm 3\sqrt{2}$.

Let the triangle have vertices $(\pm 3\sqrt{2}, -3\sqrt{2})$ (along the line $y = -3\sqrt{2}$). This triangle has a base length of $6\sqrt{2}$. Only the altitude of the triangle will affect the area, so our goal is to find the point on the circle which is farthest from the base of the triangle. Since the base of the triangle is perpendicular with the y-axis, we are looking for the point on the circle with a y-coordinate farthest from $y = -3\sqrt{2}$, which is to say that we are looking for the largest possible value of y such that $x^2 + y^2 = 36$. Since $x^2 \geq 0$, $y^2 \leq 36$ and $(0, 6)$ is the point on the circle farthest from the base of the triangle.

Now we compute the area. The base has length $6\sqrt{2}$ and the height is $6 + 3\sqrt{2}$, so the area is

$$\frac{1}{2}(6\sqrt{2})(6 + 3\sqrt{2}) = 18\sqrt{2} + 18.$$

10. We can build a recursion by noticing that for $n \geq 2$, Jon can walk up n stairs either by walking up $n-2$ then taking the last 2 in one step, or by walking up $n-1$ stairs and stepping up the last step. If G_n is the number of ways in which Jon can walk up n steps, we have

$$G_n = G_{n-1} + G_{n-2}.$$

Since there is 1 way in which Jon can walk up no stairs and 1 way in which Jon can walk up 1 step, we have $G_0 = G_1 = 1$ and the recursion produces Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, $\boxed{610}$. $G_{14} = \boxed{610}$ is our answer.

11. The biggest difficulty in solving a problem like this lies in constructing the diagram. If we extend lines AF and CD to meet at point G, then $ABCG$ is a rectangle (three right angles means the fourth must be right as well). Next, note that

$$\angle E = 360^\circ - (\angle G + \angle D + \angle F) = 360^\circ - (90^\circ + 80^\circ + 100^\circ) = 90^\circ,$$

so $\triangle FED$ is right (exterior angles of the hexagon were used for this calculation, though it doesn't change the result). We now recognize our diagram as a rectangle with a flap folded inward. We will need to subtract twice the area of the flap from the rectangle to get the area of the hexagon.

Since $AB = 7$ and $BC = 10$, $[ABCG] = 7 \cdot 10 = 70$. The area of the hexagon is 70 minus twice the area of $\triangle FED$. A system of equations helps us determine the lengths of the legs of this triangle:

$$\begin{aligned} EF + DE &= 12 \\ EF - DE &= BC - AB = 10 - 7 = 3 \end{aligned}$$

Since the legs have lengths $\frac{15}{2}$ and $\frac{9}{2}$,

$$[FED] = \frac{1}{2} \cdot \frac{15}{2} \cdot \frac{9}{2} = \frac{135}{8}.$$

Finally, the area of our hexagon is

$$[ABCG] - ([FED] + [FGD]) = [ABCG] - 2[FED] = 70 - \frac{135}{4} = \frac{145}{4} = \boxed{36.25}.$$

12. We can first simplify the problem by letting $x_1 = a-1$, $x_2 = b-1$, $x_3 = c-1$, and $x_4 = d-1$, such that $x_1 + x_2 + x_3 + x_4 = 8$.

We can certainly solve this problem without such a simplification, but it allows us to apply combinatorics more easily.

We can now model the problem in terms of Republicans and Democrats. Imagine a group of 11 politicians in which 8 are Republicans and 3 are Democrats. All the Republicans are identical and all the Democrats are identical, so the number of ways in which we can seat them in a row of 11 chairs is

$$\frac{11!}{3!8!} = \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} = 165.$$

Now, imagine that you walk down this row of politicians and begin counting the number of Republicans. Each time you get to a Democrat, you write down the number of Republicans that you counted on the way, and then start over again. You also write down the number of Republicans you have counted after the last Democrat when you get to the end of the row. When you are done, you will have written down four whole numbers that sum to 8. Each possible arrangement of the politicians is therefore in one-to-one correspondence with the solutions (x_1, x_2, x_3, x_4), so the answer is $\boxed{165}$.

13. From the formula for the number of positive divisors of a natural number, we know that we are looking for a natural number in the form p^5 or p^2q for primes p and q.

In the first case, we note that

$$1 + \frac{1}{p} + \cdots + \frac{1}{p^5} < 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2,$$

so there are no solutions for this case.

We are looking for $N = p^2q$ such that

$$1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{q} + \frac{1}{pq} + \frac{1}{p^2q} = 2.$$

Factorization is similar to that for the sum of the positive divisors:

$$\left(1 + \frac{1}{p} + \frac{1}{p^2}\right)\left(1 + \frac{1}{q}\right) = 2,$$

thus

$$\left(\frac{p^2 + p + 1}{p^2}\right)\left(\frac{q + 1}{q}\right) = 2.$$

Let's get rid of the ugliness of the fractions by multiplying through by p^2q:

$$(p^2 + p + 1)(q + 1) = 2p^2q.$$

Some students might at this point recognize that the left hand side of this equation is the sum of the divisors of p^2q and the right hand side is twice p^2q. This means that p^2q is a perfect number. It's fine not to notice such a piece of trivia, though it might help in locating the solution more quickly as there is only one possible perfect number with that kind of prime factorization.

Now we can compare the factorizations of both sides of the equation to see if we can find a clue as to how to work with it. By the Euclidean Algorithm, $(p^2+p+1, p) = (p(p+1)+1, p) = (1, p) = 1$ and $(q + 1, q) = 1$. Since p^2 doesn't divide $p^2 + p + 1$, it must divide $q + 1$. Since q doesn't divide $q + 1$, it must divide $p^2 + p + 1$.

This means we can divide both sides by p^2 and q in such a way as to produce two algebraic fractions that represent integers whose product is 2:

$$\left(\frac{p^2 + p + 1}{q}\right)\left(\frac{q + 1}{p^2}\right) = 2.$$

Since the only ways in which this product can break down is $1 \cdot 2 = 2$ or $2 \cdot 1 = 2$, we have two possible systems of equations:

$$\begin{aligned} p^2 + p + 1 &= q \\ q + 1 &= 2p^2 \end{aligned}$$

or

$$\begin{aligned} p^2+p+1 &= 2q \\ q+1 &= p^2 \end{aligned}$$

In each equation we can substitute for q in the first equation giving the following quadratic equations:

$$\begin{aligned} p^2+p+1 &= 2p^2-1 \\ p^2+p+1 &= 2p^2-2 \end{aligned}$$

Only the first has a solution that is a positive integer, $p = 2$. Plugging in we see $q = 7$, so $p^2q = \boxed{28}$.

14. There are several possible approaches to solving Pell's equations like this one. The most common approach involves generating solutions recursively. First, notice the factorization

$$x^2 - 2y^2 = (x + \sqrt{2}y)(x - \sqrt{2}y).$$

It might surprise students at first that a factorization over irrationals would help with a number theory problem, but it helps us identify recriprocal conjugates, which in term help us generate one solution from another.

By inspection we find that (3, 2) is the solution with the smallest possible value of $x + y$. If $(x_1, y_1) = (3, 2)$, then

$$x_4 + y_4\sqrt{2} = (3 + 2\sqrt{2})^4 = 577 + 408\sqrt{2}$$

gives us the fourth smallest ordered pair of solutions. Our answer is $577 + 408 = \boxed{985}$.

15. We could solve this problem with an enormous amount of case-work or an extraordinarily ugly application of variables. However, this looks like a job for generating functions!

The generating function for the roll of a single 8-sided die is

$$f(x) = x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8,$$

so the generating function for the roll of a pair of dice is

$$f^2(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^2.$$

Let the generating functions for Montana's dice be $p(x)$ and $g(x)$. Since the probability of any sum is the same for Montana's dice as a normal pair of dice, we know that

$$f^2(x) = p(x)g(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^2.$$

We are told that the largest number on the face of either of Montana's dice is 11, so we can assume without loss of generality that the degree of $p(x)$ is 11. We are now hunting for an alternate factorization for the sum of two dice:

$$\begin{aligned} p(x)g(x) &= (x+x^2+x^3+x^4+x^5+x^6+x^7+x^8)^2 \\ &= x^2(1+x+x^2+x^3+x^4+x^5+x^6+x^7)^2 \\ &= x^2\left(\frac{x^8-1}{x-1}\right)^2 \\ &= x^2\left(\frac{(x^4+1)(x^4-1)}{x-1}\right)^2 \\ &= x^2\left(\frac{(x^4+1)(x^2+1)(x^2-1)}{x-1}\right)^2 \\ &= x^2\left(\frac{(x^4+1)(x^2+1)(x+1)(x-1)}{x-1}\right)^2 \\ &= x^2(x^4+1)^2(x^2+1)^2(x+1)^2 \end{aligned}$$

Now, consider what we know about the polynomial function of a particular die. There are 8 sides, so the sum of the coefficients must be 8. This means that $p(1) = g(1) = 8$. Also, each coefficient must also be nonnegative because you can't have a negative number of faces representing a particular number. Finally, neither polynomial can have a constant because the numbers on the faces must be natural numbers, and a constant has a degree of 0, which is not a natural number. This means that x must be a factor of each p and g.

The sums of the coefficients of the individual (unsquared) factors are 1, 2, 2, and 2 (as ordered in the last factorization). In order for $p(1) = g(1) = 8$, we must split up the factors with sums of 2. The sum of the degrees of the polynomials must be 16 and p has degree 11, so g has degree 5. The only way to get a degree of 5 including a factor of x is

$$g(x) = x(x+1)^2(x^2+1) = x + 2x^2 + 2x^3 + 2x^4 + x^5,$$

thus

$$p(x) = x(x^2+1)(x^4+1)^2 = x + x^3 + 2x^5 + 2x^7 + x^9 + x^{11}.$$

One of Montana's dice has faces numbered 1, 2, 2, 3, 3, 4, 4, and 5. The other has faces numbered 1, 3, 5, 5, 7, 7, 9, and 11. The sum we seek is

$$1+2+2+3+3+4+4+5 = \boxed{24}.$$