

The goal of this solutions package is not so much to present a proof of each answer, but to try to motivate the methods and techniques used in solving them. For this reason, the solutions are longer than is necessary, but hopefully more beneficial to students.

1. First, we note that the number of terms in the LHS of the inequality is the number of terms in the arithmetic sequence $4, 6, 8, \dots, 2006$, which is $(2006 - 4)/2 + 1 = 1002$.

Now, in order for the product of the terms to be negative, an odd number of the factors must be negative, and none of them can be 0. For instance, if $x = 2005$, exactly 1 term is negative, while none are 0. In fact, 0 must be greater than an odd number of the terms. Thus, there are $1002/2 = \boxed{501}$ solutions in integers.

2. Since the variables are independent, we can break the double sum into a product of single sums:

$$\sum_{k=1}^5 \sum_{n=1}^6 kn = \sum_{k=1}^5 k \sum_{n=1}^6 n = \left(\frac{5 \cdot 6}{2}\right) \left(\frac{6 \cdot 7}{2}\right) = 15 \cdot 21 = \boxed{315}.$$

3. There is a $\frac{1}{2}$ probability that the 2 River didn't draw is the 2 of spades or the 2 of clubs. In this case, the probability she drew a card matching the suit of the 2 still in the deck (that isn't itself a 2) as her non-2 card is $\frac{12}{49}$. This gives a probability of $\frac{1}{2} \cdot \frac{12}{49} = \frac{6}{49}$ that she still wound up with exactly one spade and one club.

There is a $\frac{1}{2}$ probability that River drew both the 2 of spades and the 2 of clubs. The probability the non-2 she drew was not a spade or club is $\frac{24}{49}$ (12 diamonds and 12 hearts that aren't 2's). This case has a probability of $\frac{1}{2} \cdot \frac{24}{49} = \frac{12}{49}$.

Adding both cases, the probability is $\frac{6}{49} + \frac{12}{49} = \boxed{\frac{18}{49}}$.

4. The first digit must be at least 1, so none of the digits can be 0. For each possible set of digits, there is exactly one order in which they can occur. This means the number of such six-digit positive integers is the number of ways to choose 6 digits out of 9 possible, which is $\binom{9}{6} = (9 \cdot 8 \cdot 7)/6 = \boxed{84}$.

5. We can manipulate the LHS of the equation into a single logarithm:

$$\begin{aligned} A \log_{1200} 2 + B \log_{1200} 3 + C \log_{1200} 5 &= \log_{1200} 2^A + \log_{1200} 3^B + \log_{1200} 5^C \\ &= \log_{1200} 2^A \cdot 3^B \cdot 5^C. \end{aligned}$$

Since this single logarithm equals D , we have

$$1200^D = 2^A \cdot 3^B \cdot 5^C.$$

The RHS of our simplified equation is a prime factorization. It makes sense to turn the LHS into a prime factorization in order to compare apples to apples:

$$1200^D = (2^4 \cdot 3^1 \cdot 5^2)^D = 2^{4D} \cdot 3^D \cdot 5^{2D} = 2^A \cdot 3^B \cdot 5^C.$$

Since all the variables are integers, these prime factorizations match up term-by-term, giving us a system of equations:

$$\begin{aligned}4D &= A, \\ D &= B, \\ 2D &= C.\end{aligned}$$

In terms of D , we have $(A, B, C, D) = (4D, D, 2D, D)$. Since the variables share no common factor greater than 1, the value of D cannot be greater than 1 (because D is a divisor of all four integers). Thus, $A + B + C + D = 4 + 1 + 2 + 1 = \boxed{8}$.

6. Adding all three equations gives

$$x + \lfloor x \rfloor + \{x\} + y + \lfloor y \rfloor + \{y\} + z + \lfloor z \rfloor + \{z\} = 2x + 2y + 2z = 45.6,$$

thus $10(x + y + z) = 5 \cdot 45.6 = \boxed{228}$.

7. Consider a diagonal of the larger square. It's length is equal to the sum of the altitudes of two of the equilateral triangles and one side length of the smaller square. If d is this diagonal length, then the area of the large square is

$$\frac{d^2}{2} = \frac{(10\sqrt{3} + 10)^2}{2} = \boxed{200 + 100\sqrt{3}}.$$

8. Consider any sequence of modulo 7 residues, starting with 0, such that each is the sum of the previous two. If the second is a , then the next few are

$$\begin{aligned}0 + a &\equiv a \pmod{7} \\ a + a &\equiv 2a \pmod{7} \\ a + 2a &\equiv 3a \pmod{7} \\ 2a + 3a &\equiv 5a \pmod{7} \\ 3a + 5a &\equiv a \pmod{7} \\ 5a + a &\equiv 6a \pmod{7} \\ a + 6a &\equiv 0 \pmod{7}\end{aligned}$$

So, every 8th term is a multiple of 7. This means $1000/8 = 125$ of all 1000 terms are multiples of 7, so $125 - 1 = \boxed{124}$ of the other 999 terms are multiples of 7.

9. We know that $a + b + c = 12$. Let $x = a - 1$, $y = b - 1$, and $z = c - 1$. So, we have $x + y + z = 9$, with all variables nonnegative. The number of ordered triplets (a, b, c) is the number of ordered triplets (x, y, z) .

Imagine a row of 9 chihuahuas. Two barriers are placed between or around the chihuahuas. Any chihuahuas before the first barrier represent extra chocolate donuts (more than the first one). The chihuahuas between the two barriers represent the extra glazed donuts. The chihuahuas after both barriers represent the extra powdered donuts. There are

$$\binom{11}{2} = \frac{11 \cdot 10}{2} = \boxed{55}$$

ways to arrange the barriers and chihuahua's corresponding to the number of ordered triplets (x, y, z) (and therefore (a, b, c)).

10. Setting the two probabilities equal, we have

$$\binom{2006}{1800} p^{1800} (1-p)^{206} = \binom{2006}{1801} p^{1801} (1-p)^{205}$$

$$\Leftrightarrow \frac{2006!}{1800!206!} (1-p) = \frac{2006!}{1801!205!} p \Leftrightarrow 1801(1-p) = 206p.$$

So, $1801 - 1801p = 206p \Rightarrow 1801 = 2007p \Rightarrow p = \boxed{\frac{1801}{2007}}$.

11. Note that for a nonnegative integer n , either 5^{k+1} has one more digit than 5^k , or else 5^k begins with the digit 1. Going from 5^0 (which has 1 digit) to 5^{2005} (which has 1402 digits), a total of 1401 digits are “added”. This is out of a total of 2005 changes. This leaves $2005 - 1401 = \boxed{604}$ powers of 5 that begin with the digit 1.
12. Yoda cannot select 5 of the composite integers without selecting 2 with a common prime divisor. If he could, then he could select 5 such pairwise relatively prime composites, then he would write a composite that is the multiple of a prime at least as large as the fifth smallest prime, which is 11. The smallest composite multiple of 11 is 121. The set with all relatively prime pairs could then simple include the squares of the five smallest primes. However, Yoda counted up to at most $\boxed{120}$.
13. Let $a = \sqrt[4]{x+27}$ and $b = \sqrt[4]{55-x}$, so $a+b=4$ and $a^4+b^4=82$.
Let $p = ab$. Then

$$\begin{aligned} a^2 + 2ab + b^2 &= 16 \\ \Rightarrow a^2 + b^2 &= 16 - 2p \\ \Rightarrow a^4 + 2a^2b^2 + b^4 &= 256 - 64p + p^2 = 82 + 2p^2 \\ \Rightarrow p^2 - 32p + 87 &= (p-3)(p-29) = 0. \end{aligned}$$

By AM-GM, $p = ab \leq (a+b)^2/4 = 4$, so $p = 3$. Then the solutions are $(a, b) = (1, 3)$ and $(3, 1)$, which leads to $x = -26$ and 54 . The sum is $\boxed{28}$.

14. We might start out looking for similarity between the two equations, and we can certainly find some:

$$\begin{aligned} x(x^2 - 3y^2) &= -610, \\ y(3x^2 - y^2) &= 182. \end{aligned}$$

We can factor a bit, and the coefficients are the same. We could even try to factor differences of squares into linear factors with some irrational coefficients, but that doesn't seem to get us anywhere nice.

Exploring further, we could note that adding the equations results in an equation that is close to something nice:

$$x^3 + 3x^2y - 3xy^2 - y^3 = -428.$$

With a couple of signs changed, we'd have the expansion of $(x+y)^3$ or $(x-y)^3$.

When your first idea comes close to changing a problem into something you can solve, look for some way to alter your idea to step around the remaining difficulties.

Being so close to familiar expansions, we look for some way to match what we have to one of those forms. Noting that sign changes for each term with at least 2 powers of y leads us to thinking about complex numbers: $(yi)^2 = -y^2$. Indeed, the expansion of $(x + yi)^3$ provides us with the pattern of signs we're looking for:

$$(x + yi)^3 = x^3 + x^2yi - xy^2 - y^3i.$$

Now we recognize that the system we're working with can be altered slightly to form the system we'd rather be working with:

$$\begin{aligned}x^3 - 3xy^2 &= -610, \\3x^2yi - y^3i &= 182i.\end{aligned}$$

Adding the new equations gives us

$$(x + yi)^3 = -610 + 182i.$$

Now that we're working with complex numbers, we can examine the magnitudes of each side of our last equation:

$$|(x+yi)^3| = |-610+182i| \Rightarrow \sqrt{(x^2+y^2)^3} = \sqrt{610^2 + 182^2} = \sqrt{372100 + 33124} = \sqrt{405224}.$$

Factoring 405224 might give us some insight: $405224 = 2^3 \cdot 37^2$. So, we know that $x^2 + y^2 = 74$.

Looking for a way to make use of our new information, we look to the factored form of the first equation. We can substitute for y^2 to create an equation involving just x :

$$x(x^2 - 3y^2) = x[4x^2 - 3(x^2 + y^2)] = x(4x^2 - 222) = -610.$$

Dividing by 2, expanding the equation, and rearranging, we have

$$2x^3 - 111x + 305 = 0.$$

Hunting for rational roots, we find $x = 5$ is a solution. The remaining quadratic has complex roots.

Since $x = 5$ and $x^2 + y^2 = 74$, we have $y^2 = 49$, so $y = \pm 7$. Plugging the possible values of x and y into either of our original equations, we find that $\boxed{(5, 7)}$ is the sole real solution to the original system.

15. For any given trip, Ying either starts from Viavesta, or she has a $1/3$ chance of traveling to Viavesta. Let p_n be the probability Ying is at Viavesta after n trips. We have the recursion

$$p_{n+1} = \frac{1}{3}(1 - p_n).$$

Since $p_0 = 1$, we have

$$p_1 = \frac{1}{3}(1 - p_0) = \frac{1}{3}(1 - 1) = 0,$$

$$p_2 = \frac{1}{3}(1 - p_1) = \frac{1}{3}(1 - 0) = \frac{1}{3},$$

$$p_3 = \frac{1}{3}(1 - p_2) = \frac{1}{3}\left(1 - \frac{1}{3}\right) = \frac{2}{9},$$

$$p_4 = \frac{1}{3}(1 - p_3) = \frac{1}{3}\left(1 - \frac{2}{9}\right) = \frac{7}{27},$$

$$p_5 = \frac{1}{3}(1 - p_4) = \frac{1}{3}\left(1 - \frac{7}{27}\right) = \frac{20}{81},$$

$$p_6 = \frac{1}{3}(1 - p_5) = \frac{1}{3}\left(1 - \frac{20}{81}\right) = \frac{61}{243},$$

$$p_7 = \frac{1}{3}(1 - p_6) = \frac{1}{3}\left(1 - \frac{61}{243}\right) = \frac{182}{729},$$

$$p_8 = \frac{1}{3}(1 - p_7) = \frac{1}{3}\left(1 - \frac{182}{729}\right) = \frac{547}{2187}.$$

Since the cases are symmetric, Ying has equal probabilities of ending up at any of the other cities, so $q = (1 - 547/2187)/3 = 1640/6561$. Thus, $p + q = p_8 + q = \boxed{\frac{3281}{6561}}$.

Some of the problems and solutions are excerpted from the upcoming Art of Problem Solving text *Intermediate Algebra*.