The 1st Annual iTest Tournament of Champions

Round 1 Problems and Solutions

1. Find the remainder when 3^{2007} is divided by 2007.

   Answer: 1620

   Solution: Since $3^2 \cdot 223^1$ is the prime factorization of 2007, we can find the modulo 2007 residue of a number from its modulo 9 and modulo 223 residues.

   Clearly $3^{2007} \equiv 0 \pmod{9}$. By Fermat's Little Theorem, $3^{222} \equiv 1 \pmod{223}$. So,

$$\begin{aligned} 3^{2007} &\equiv \left(3^{222}\right)^9 \cdot 3^9 \equiv 1^9 \cdot 3^9 \\ &\equiv 3^9 \equiv 3^4 \cdot 3^5 \\ &\equiv 81 \cdot 243 \equiv 81 \cdot 20 \\ &\equiv 1620 \pmod{223}. \end{aligned}$$

   Since $9 \mid 1620$, we have our answer. Otherwise, we could note that $1620 \equiv 59 \pmod{223}$ and solve the system of linear congruences

$$\begin{aligned} r &\equiv 0 \pmod{9}, \\ r &\equiv 59 \pmod{223}. \end{aligned}$$

2. Let a/b be the probability that a randomly chosen positive divisor of 12^{2007} is also a divisor of 12^{2000}, where a and b are relatively prime positive integers. Find the remainder when $a+b$ is divided by 2007.

   Answer: 79

   Solution: The number of positive divisors of

$$12^{2007} = 2^{4014} \cdot 3^{2007}$$

   is $4015 \cdot 2008$. The number of positive divisors of

$$12^{2000} = 2^{4000} \cdot 3^{2000}$$

   is $4001 \cdot 2001$, and all these are divisors of 12^{2007}. Thus, the probability that a positive divisor of 12^{2007} is also a divisor of 12^{2000} is equal to

$$\frac{4001 \cdot 2001}{4015 \cdot 2008}.$$

   Now, in order to see whether or not this fraction reduces, we note each of the following:

$$\begin{aligned} \gcd(4015, 4001) &= \gcd(4015, 14) &&= \gcd(11, 14) &&= 1, \\ \gcd(4015, 2001) &= \gcd(4015, 13) &&= \gcd(11, 13) &&= 1, \\ \gcd(2008, 4001) &= \gcd(2008, -15) &&= \gcd(13, -15) &&= 1, \\ \gcd(2008, 2001) &= \gcd(7, 2001) &&= \gcd(7, 6) &&= 1. \end{aligned}$$

   Thus, when we multiply out the numerator and denominator of our fraction above (the probability), the resulting fraction does not reduce. So, $a = 4015 \cdot 2008$ and $b = 4001 \cdot 2001$.

   Now we compute

$$\begin{aligned} 4015 \cdot 2008 + 4001 \cdot 2001 &\equiv 1 \cdot 1 + (-13) \cdot (-6) \\ &\equiv 1 + 78 \equiv 79 \pmod{2007}. \end{aligned}$$

3. For each positive integer n, let $g(n)$ be the sum of the digits when n is written in binary. For how many positive integers n, where $1 \leq n \leq 2007$, is $g(n) \geq 3$?

   Answer: 1941

   Solution: We approach this problem using complementary counting. It's easier to count the integers n from 1 to 2007 inclusive such that $g(n) < 3$. We have just the cases $g(n) = 1$ and $g(n) = 2$.

   - If $g(n) = 1$, then n has a binary representation that is a 1 followed by some number of zeros. In other words, n is a pure power of 2. Thus, $g(n) = 1$ for $2^0 = 1, 2^1 = 2, 2^2 = 4, \ldots, 2^9 = 512, 2^{10} = 1024$, which is 11 values.
   - If $g(n) = 2$, then n is the sum of two distinct pure powers of 2. While counting these, we note that $2^{10} < 2007 < 2^{11}$ and also that $2^9 + 2^{10} < 2007$. This means $g(n) = 2$ for exactly the integers $n = 2^a + 2^b$, where (a, b) are distinct nonnegative integers no greater than 10, and $a > b$. There are $\binom{11}{2} = 55$ such integers n.

   In total, there are $11 + 55 = 66$ integers n such that $1 \leq n \leq 2007$ and $g(n) < 3$. This leaves $2007 - 66 = 1941$ integers in that range such that $g(n) \geq 3$.

4. Black and white coins are placed on some of the squares of a 418×418 grid. All black coins that are in the same row as any white coin(s) are removed. After that, all white coins that are in the same column as any black coin(s) are removed. If b is the number of black coins remaining and w is the number of remaining white coins, find the remainder when the maximum possible value of bw gets divided by 2007.

   Answer: 952

   Credit: This problem was inspired by a problem from the 2000 Russian Mathematical Olympiad.

   Solution: After both "operations" on the grid, each row and column contains at most one color of coin (black or white). Let m be the number of rows in which there are black coins and let n be the number of columns in which there are black coins. Then $b \leq mn$. Also, $418 - m$ and $418 - n$ are the numbers of rows and columns, respectively, in which there are white coins. Thus, $w \leq (418 - m)(418 - n)$. Hence

$$bw \leq mn(418 - m)(418 - n) = m(418 - m)n(418 - n).$$

   Now, by AM-GM,

$$\sqrt{m(418 - m)} \leq \frac{m + (418 - m)}{2} = 209 \qquad \Rightarrow \qquad m(418 - m) \leq 209^2,$$

   with equality iff $m = 209$. Likewise, $n(418 - n) \leq 209^2$ with equality iff $n = 209$. Thus,

$$bw \leq m(418 - m)n(418 - n) \leq 209^4,$$

   with equality in the right inequality iff $m = n = 209$. In particular, we maximize bw by completely filling 209×209 grids, one with only black coins and one with only white coins, then placing the two grids corner-to-corner so that they fill up half a 418×418 grid. So, the maximum possible value of bw is 209^4.

   Now we note that

$$\begin{aligned} 209^4 &\equiv (209^2)^2 \equiv 43681^2 \\ &\equiv 1534^2 \equiv (-1534)^2 \equiv 473^2 \\ &\equiv 23029 \equiv 952 \pmod{2007}. \end{aligned}$$

5. Find the largest possible value of $a+b$, less than or equal to 2007, for which a and b are relatively prime, and such that there is some positive integer n for which

$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\frac{4^3-1}{4^3+1}\cdots\frac{n^3-1}{n^3+1}=\frac{a}{b}.$$

Answer: 1891

Solution: There are a number of ways to discover that the product telescopes, though a straight-forward approach to evaluating the product involves manipulation of the product via indices while using product notation:

$$\prod_{k=2}^{n}\frac{k^3-1}{k^3+1} = \frac{\prod\limits_{k=2}^{n}k^3-1}{\prod\limits_{k=2}^{n}k^3+1} = \frac{\left(\prod\limits_{k=2}^{n}k-1\right)\left(\prod\limits_{k=2}^{n}k^2+k+1\right)}{\left(\prod\limits_{k=2}^{n}k+1\right)\left(\prod\limits_{k=2}^{n}k^2-k+1\right)}$$

$$= \frac{\left(\prod\limits_{k=1}^{n-1}k\right)\left(\prod\limits_{k=2}^{n}k^2+k+1\right)}{\left(\prod\limits_{k=3}^{n+1}k\right)\left(\prod\limits_{k=1}^{n-1}k^2+k+1\right)} = \frac{1\cdot 2(n^2+n+1)}{n(n+1)(2^2-2+1)} = \frac{2(n^2+n+1)}{3(n^2+n)}.$$

Now we have that

$$\frac{2(n^2+n+1)}{3(n^2+n)}=\frac{a}{b},$$

and in order to examine possible values of $a+b$, we must determine exactly when and how the fraction on the left reduces. From the Euclidean Algorithm for Polynomials, n^2+n+1 and n^2+n have no common factors greater than 1. So, the only possible factors that could cancel are a single power of 2 and a single power of 3.

Since either n or $n+1$ is even, we know that $n^2+n=n(n+1)$ is divisible by 2, so we can always reduce the numerator and denominator by a factor of 2. However, n^2+n+1 is divisible by 3 iff $n\equiv 1 \pmod 3$. This leaves us with two cases:

- When $n\equiv 1 \pmod 3$, then

$$a+b=\frac{2(n^2+n+1)}{6}+\frac{3(n^2+n)}{6}=\frac{5n^2+5n+2}{6},$$

- When $n\not\equiv 1 \pmod 3$, then

$$a+b=\frac{2(n^2+n+1)}{2}+\frac{3(n^2+n)}{2}=\frac{5n^2+5n+2}{2}.$$

In the first case, we need to maximize n such that

$$\frac{5n^2+5n+2}{6}\le 2007 \qquad\Leftrightarrow\qquad n(n+1)\le 2408.$$

Since $49 = \sqrt{2401} < \sqrt{2408} < \sqrt{2500} = 50$, we see quickly that $n < 49$. Since $n \equiv 1 \pmod 3$, the largest such n is 46. When $n = 46$,

$$a + b = \frac{5n^2 + 5n + 2}{6} = \frac{5n(n+1) + 2}{6} = \frac{5 \cdot 46 \cdot 47 + 2}{6} = 1802.$$

In the second case, we need to maximize n such that

$$\frac{5n^2 + 5n + 2}{2} \leq 2007 \qquad \Leftrightarrow \qquad n(n+1) \leq 802.4.$$

Plugging in values for n just below 30, we find that $n \leq 27$. In fact, $n = 27 \not\equiv 1 \pmod 3$, so the maximum possible $a + b$ in this case is

$$a + b = \frac{5n^2 + 5n + 2}{2} = \frac{5 \cdot 27 \cdot 28 + 2}{2} = 1891.$$

Between the two cases, the largest possible value of $a + b$ less than or equal to 2007 is 1891.

Round 2 Problems and Solutions

1. Let a and b be perfect squares whose product exceeds their sum by 4844. Compute the value of

$$\left(\sqrt{a}+1\right)\left(\sqrt{b}+1\right)\left(\sqrt{a}-1\right)\left(\sqrt{b}-1\right)-\left(\sqrt{68}+1\right)\left(\sqrt{63}+1\right)\left(\sqrt{63}-1\right)\left(\sqrt{68}-1\right).$$

   Answer: 691

   Solution: We are given that

$$ab-(a+b)=4844 \qquad \Leftrightarrow \qquad ab-a-b+1=4845,$$

   by Simon's Favorite Factoring Trick. Factoring the LHS of the last equation, we see that

$$\begin{aligned} ab-a-b+1 &= (a-1)(b-1) \\ &= \left(\sqrt{a}+1\right)\left(\sqrt{a}-1\right)\left(\sqrt{b}+1\right)\left(\sqrt{b}-1\right) \\ &= \left(\sqrt{a}+1\right)\left(\sqrt{b}+1\right)\left(\sqrt{a}-1\right)\left(\sqrt{b}-1\right). \end{aligned}$$

   Thus, we see that

$$\begin{aligned} & \left(\sqrt{a}+1\right)\left(\sqrt{b}+1\right)\left(\sqrt{a}-1\right)\left(\sqrt{b}-1\right)-\left(\sqrt{68}+1\right)\left(\sqrt{63}+1\right)\left(\sqrt{63}-1\right)\left(\sqrt{68}-1\right) \\ = {} & (ab-a-b+1)-(68-1)(63-1) \\ = {} & 4845-67\cdot 62=4845-4154=691. \end{aligned}$$

2. The area of triangle ABC is 2007. One of its sides has length 18, and the tangent of the angle opposite that side is 2007/24832. When the altitude is dropped to the side of length 18, it cuts that side into two segments. Find the sum of the squares of those two segments.

   Answer: 194

   Solution: Note that the altitude to the side of length 18 is $2\cdot 2007/18=223$. This altitude divides the angle from the vertex opposite the side into angles we'll call x and y. Then

$$\begin{aligned} \tan x &= \frac{a}{223}, \\ \tan y &= \frac{b}{223}, \\ \tan(x+y) &= \frac{2007}{24832}, \end{aligned}$$

   where a and b are the lengths of the segments that make up the side of length 18.

   Now, using the tangent addition identity, we have two expressions for $\tan(x+y)$:

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{\frac{a}{223}+\frac{b}{223}}{1-\frac{ab}{223^2}}=\frac{223(a+b)}{223^2-ab}=\frac{223\cdot 18}{223^2-ab}=\frac{4014}{223^2-ab}=\frac{2007}{24832}.$$

   Thus, $223^2-ab=2\cdot 24832=49664$, hence $ab=65$. So,

$$a^2+b^2=(a+b)^2-2ab=18^2-2(65)=324-130=194.$$

   We could also have found that a and b are 5 and 13 in some order.

3. Find the smallest value of n for which the series

$$1 \cdot 3^1 + 2 \cdot 3^2 + 3 \cdot 3^3 + \cdots + n \cdot 3^n$$

exceeds 3^{2007}.

Answer: 2000

Solution: If we can find a closed form for the sum (of this arithmetico-geometric series), we might be able to compare that formula to the value of 3^{2007}. Rewriting the series in summation notation helps us see the technique more easily:

$$\begin{aligned}
\sum_{k=1}^{n} k3^k &= (3^1) + (3^2 + 3^2) + (3^3 + 3^3 + 3^3) + \cdots + (3^n + 3^n + \cdots + 3^n) \\
&= (3^1 + 3^2 + 3^3 + \cdots + 3^n) + (3^2 + 3^3 + \cdots + 3^n) + \cdots + (3^{n-1} + 3^n) + (3^n) \\
&= \sum_{k=1}^{n} 3^k + \sum_{k=2}^{n} 3^k + \cdots + \sum_{k=n}^{n} 3^k \\
&= \sum_{j=1}^{n}\sum_{k=j}^{n} 3^k.
\end{aligned}$$

The inner sum is a geometric series that we can evaluate as a closed expression:

$$\sum_{k=j}^{n} 3^k = \frac{3^{n+1} - 3^j}{2}.$$

Thus,

$$\sum_{j=1}^{n}\sum_{k=j}^{n} 3^k = \sum_{j=1}^{n} \frac{3^{n+1} - 3^j}{2}.$$

We can break this sum into two pieces:

$$\begin{aligned}
\sum_{j=1}^{n} \frac{3^{n+1} - 3^j}{2} &= \frac{1}{2}\sum_{j=1}^{n} \left(3^{n+1} - 3^j\right) \\
&= \frac{1}{2}\sum_{j=1}^{n} 3^{n+1} - \frac{1}{2}\sum_{j=1}^{n} 3^j.
\end{aligned}$$

The first piece is simple to evaluate since the index variable is not part of the summed expression:

$$\frac{1}{2}\sum_{j=1}^{n} 3^{n+1} = \frac{1}{2} \cdot n \cdot 3^{n+1} = \frac{n3^{n+1}}{2}.$$

The second piece is another geometric series:

$$\frac{1}{2}\sum_{j=1}^{n} 3^j = \frac{1}{2} \cdot \frac{3^{n+1} - 3}{2} = \frac{3^{n+1} - 3}{4}.$$

So,

$$\frac{1}{2}\sum_{j=1}^{n} 3^{n+1} - \frac{1}{2}\sum_{j=1}^{n} 3^{j} = \frac{n3^{n+1}}{2} - \frac{3^{n+1}-3}{4} = \frac{(2n-1)3^{n+1}+3}{4}.$$

Now, our goal is to find the minimum value of n such that

$$\frac{(2n-1)3^{n+1}+3}{4} > 3^{2007}.$$

The quick estimate $n = 2000$ seems fairly close to equality, so we start there to see how close we are:

$$\begin{aligned}(2\cdot 2000-1)3^{2001}+3 &> 3999\cdot 3^{2001}\\ &> 4\cdot 729\cdot 3^{2001} = 4\cdot 3^{2007}.\end{aligned}$$

A similar quick check for $n = 1999$ shows that $n = 2000$ is in fact the smallest possible n.

4. For each positive integer n, let $S_n = \sum\limits_{k=1}^{n} k^3$, and let $d(n)$ be the number of positive divisors of n. For how many positive integers m, where $m \leq 25$, is there a solution n to the equation $d(S_n) = m$?

Answer: 6

Solution: First, we can find a closed form expression for S_n:

$$S_n = \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right].$$

Now, we consider what this tells us about the possible values of $d(S_n)$. Since S_n is a perfect square, the value of m must be odd. We also note for $n > 2$, that S_n is the square of a composite number, because

$$\frac{n(n+1)}{2} = \frac{n}{2}\cdot(n+1) = n\cdot\frac{n+1}{2},$$

where one of the two products is the product of two integers. Since $\gcd(n, n+1) = 1$, these integers are in fact relatively prime, so the prime factorization of S_n includes at least two primes for $n > 2$. So, when we compute $d(S_n)$, adding 1 to each of the exponents of the prime factorization of S_n, then taking the product of the results, we wind up with a composite number (an odd composite number in particular).

Now we take care of the special cases where $n \leq 2$. We note that $d(S_1) = d(1) = 1$ and that $d(S_2) = d(9) = 3$. So, there are solutions to the given equation when $m = 1$ or $m = 3$. Next, we consider which odd composites can be values of m and yield solutions. Just checking a few values of n, we find that

$$\begin{aligned}d(S_3) = d(6^2) &= d(2^2\cdot 3^2) = (2+1)(2+1) = 9,\\ d(S_7) = d(28^2) &= d(2^4\cdot 7^2) = (4+1)(2+1) = 15,\\ d(S_8) = d(36^2) &= d(2^4\cdot 3^4) = (4+1)(4+1) = 25.\end{aligned}$$

Many solvers probably used more creative means to hunt down values of n to match the possible odd composite values of m.

The only possible odd composite left in the range of m is $m = 21$. We need to find an n such that $d(S_n) = 21$, or show that no such value exists. When $d(S_n) = 21$, we know that

$$S_n = (p^3 \cdot q^1)^2,$$

for some primes p and q. Looking to construct an example, we note that $2^4 + 1$ is prime, so

$$S_{16} = \left(2^3 \cdot 17^1\right)^2 \qquad \Rightarrow \qquad d(S_{16}) = 21.$$

Finally, we note that there are 6 possible values of m (1, 3, 9, 15, 21, 25) where $m \leq 25$ and there is a solution to the equation $d(S_n) = m$.

5. A polynomial $p(x)$ of degree 1000 is such that $p(n) = (n+1)2^n$ for all nonnegative integers n such that $n \leq 1000$. Given that
$$p(1001) = a \cdot 2^b - c,$$
where a is an odd integer, and $0 < c < 2007$, find $c - (a + b)$.

   Answer: 500

   Solution: We employ a difference table to see what we can discover about $p(x)$:

$$\begin{array}{rccccccccccc}
P(n): & 1 & & 4 & & 12 & & 32 & & 80 & & 192 \\
\Delta P(n): & & 3 & & 8 & & 20 & & 48 & & 112 & \\
\Delta^2 P(n): & & & 5 & & 12 & & 28 & & 64 & & \\
\Delta^3 P(n): & & & & 7 & & 16 & & 36 & & & \\
\Delta^4 P(n): & & & & & 9 & & 20 & & & & \\
\Delta^5 P(n): & & & & & & 11 & & & & &
\end{array}$$

It certainly appears that the diagonal going down and right from $p(0) = 1$ runs through the odd integers (until the first constant row in the table, after which all rows are filled with 0's). Since the degree of p is 1000, the 1001st row consists of the constant entry 2001.

In order to test our hypothesis, we apply the formula for constructing a polynomial from the "zero diagonal" of a difference table. For $0 \leq n \leq 1000$, we have

$$\begin{aligned}
p(n) &= 1\binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + \cdots + (2n+1)\binom{n}{n} \\
&= \frac{1}{2}\left[1\binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + \cdots + (2n+1)\binom{n}{n}\right] \\
&\quad + \frac{1}{2}\left[(2n+1)\binom{n}{0} + (2n-1)\binom{n}{1} + (2n-3)\binom{n}{2} + \cdots + 1\binom{n}{n}\right] \\
&= \frac{1}{2}\left[(2n+2)\binom{n}{0} + (2n+2)\binom{n}{1} + (2n+2)\binom{n}{2} + \cdots + (2n+2)\binom{n}{n}\right] \\
&= (n+1)\left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}\right] = (n+1)2^n.
\end{aligned}$$

Thus, our hypothesis is correct.

Now, we compute $p(1001)$ similarly, except that the combinatorial symmetry breaks down just a little in the calculations:

$$p(1001) = 1\binom{1001}{0} + 3\binom{1001}{1} + 5\binom{1001}{2} + \cdots + 2001\binom{1001}{1000}$$

$$= 1\binom{1001}{0} + 3\binom{1001}{1} + 5\binom{1001}{2} + \cdots + 2001\binom{1001}{1000} + 2003\binom{1001}{1001} - 2003$$

$$= 1002\left[\binom{1001}{0} + \binom{1001}{1} + \binom{1001}{2} + \cdots + \binom{1001}{1001}\right] - 2003$$

$$= 1002 \cdot 2^{1001} - 2003 = 501 \cdot 2^{1002} - 2003.$$

Since $0 < c < 2007$, the form of our answer we seek must be $501 \cdot 2^{1002} - 2003$, in which case $c - (a+b) = 2003 - (501 + 1002) = 2003 - 1503 = 500$.

Round 3 Problems and Solutions

1. Let A be the area of the locus of points z in the complex plane that satisfy $|z + 12 + 9i| \leq 15$. Compute $\lfloor A \rfloor$.

   Answer: 706

   Solution: We are looking for the locus of points that satisfy $|z + 12 + 9i| \leq 15$, which is equivalent to the equation $|z - (-12 - 9i)| \leq 15$, which we interpret as the points z whose distance from the point $-12 - 9i$ is no greater than 15. Hence, the locus is a disk of radius 15 whose area is $15^2\pi = 225\pi$, which is approximately equal to

   $$225 \cdot 3.1416 = 9 \cdot 25 \cdot 3.1416 = 9 \cdot 78.54 = 706.86.$$

   This estimate is clearly suitable, so $\lfloor A \rfloor = 706$.

2. Al and Bill play a game involving a fair six-sided die. The die is rolled until either there is a number less than 5 rolled on consecutive tosses, or there is a number greater than 4 on consecutive tosses. Al wins if the last roll is a 5 or a 6. Bill wins if the last roll is a 2 or lower. Let m and n be relatively prime positive integers such that m/n is the probability that Bill wins. Find the value of $m + n$.

   Answer: 29

   Credit: This problem is adapted from one that appeared on the 1985 shortlist for the International Mathematical Olympiad.

   Solution: Some students were initially concerned about the fact that the game does not always result in either Al or Bill winning. Hopefully, this twist did not confuse or mislead any contestants.

   Denoting a roll of 1-4 as L, and a roll of 5 or 6 as H, we represent the game as a string of L's and H's where the letters alternate until one repeats. Bill can only win when LL are the last two letters. So, we find the probability that LL are the last two letters:

Sequence	Probability
LL	$\left(\frac{2}{3}\right)^2$,
HLL	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2$,
LHLL	$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3$,
HLHLL	$\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3$,
⋮	⋮

   We sum these probabilities as two geometric series:

$$\begin{aligned}
&\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 + \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 + \cdots \\
= \;& \left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 + \cdots\right] + \left[\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 + \cdots\right] \\
= \;& \left(1 + \frac{1}{3}\right)\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 + \cdots\right] \\
= \;& \frac{4}{3}\left(\frac{\frac{4}{9}}{1 - \frac{2}{9}}\right) = \frac{4}{3} \cdot \frac{4}{7} = \frac{16}{21}.
\end{aligned}$$

Bill wins half the time that the final roll is a low (L) roll, so the probability Bill wins is 8/21. The answer is thus $8 + 21 = 29$.

3. Find the largest natural number n such that

$$2^n + 2^{11} + 2^8$$

is a perfect square.

Answer: 12

Solution: One way to get started is to search first for values of n such that $n \geq 8$. This way, the given quantity is a multiple of 2^8. So, if

$$k^2 = 2^n + 2^{11} + 2^8$$

is a perfect square, then k^2 is a multiple of 2^8. Hence, $16 \mid k$, and so $k = 16m$ for some integer m, and

$$m^2 = 2^{n-8} + 2^3 + 1 = 2^{n-8} + 9.$$

Now we can equate a (factorable) difference of squares and a power of 2:

$$m^2 - 9 = 2^{n-8} \qquad \Leftrightarrow \qquad (m+3)(m-3) = 2^{n-8}.$$

Thus $m+3$ and $m-3$ are powers of 2. Their difference is 6, and 2 is the largest power of 2 that divides 6, so 2 must be $m-3$, hence $m = 5$ is the only possible value of m. Thus,

$$2^{n-8} = 5^2 - 9 = 16 \qquad \Leftrightarrow \qquad n = 12.$$

Since $n = 12$ is the only value of n such that $n \geq 8$ and for which the given expression is a perfect square, 12 is our answer. In fact, $n = 12$ is the only such natural number.

4. Find the smallest positive integer k such that

$$(16a^2 + 36b^2 + 81c^2)(81a^2 + 36b^2 + 16c^2) < k(a^2 + b^2 + c^2)^2,$$

for some ordered triple of positive integers (a, b, c).

Answer: 1297

Solution: The two series of squares remind us of the Cauchy-Schwarz Inequality, though the inequality sign is strict and not in the direction that we are accustomed to seeing it in. In fact, by Cauchy-Schwarz, we have

$$(16a^2 + 36b^2 + 81c^2)(81a^2 + 36b^2 + 16c^2) \geq (36a^2 + 36b^2 + 36c^2)^2 = 36^2(a^2 + b^2 + c^2).$$

So, the smallest k could be is $36^2 + 1 = 1297$. But the task remains to figure out if we can construct an example or if the integer requirements limit k in some way. After all, equality cannot be achieved in our inequality above because by the equality condition for Cauchy-Schwarz,

$$16a^2 = t(81a^2), \qquad 36b^2 = t(36b^2), \qquad 81c^2 = t(16c^2),$$

for some value of t. But this system of equations has no solutions in which a, b, and c are positive integers (only when $a = b = c = 0$).

Now, we look for some way to test out whether or not 1297 can be achieved for k. Holding some variables in a functional expression constant is often a good way to explore its behavior. Letting $b = c = 1$, we have that

$$\begin{aligned}(16a^2 + 36b^2 + 81c^2)(81a^2 + 36b^2 + 16c^2) &= (16a^2 + 36 + 81)(81a^2 + 36 + 16)\\ &= (16a^2 + 97)(81a^2 + 52)\\ &= 1296a^4 + 8689a^2 + 5044.\end{aligned}$$

Also,

$$1297(a^2 + 2)^2 = 1297a^4 + 5188a^2 + 5188.$$

So, we are trying to determine whether or not there is a positive integer a such that

$$1296a^4 + 8689a^2 + 5044 < 1297a^4 + 5188a^2 + 5188 \qquad \Leftrightarrow \qquad a^4 - 3501a^2 + 144 > 0,$$

which of course there is for sufficiently large values of a, and $k = 1297$ is indeed the answer.

5. Let $s = a + b + c$, where a, b, and c are integers that are lengths of the sides of a box. The volume of the box is numerically equal to the sum of the lengths of the twelve edges of the box plus its surface area. Find the sum of the possible values of s.

   Answer: 684

   Credit: A different version of this problem was proposed by K. R. S. Sastry and appeared with a similar solution below (by Murray S. Klamkin and Andy Liu) in the November 1996 publication of *The College Mathematics Journal.*

   Solution: We begin by writing the information about the box in terms of a, b, and c:

$$abc = 4(a + b + c) + 2(bc + ca + ab). \tag{1}$$

Often we use factored algebraic expressions to help solve Diophantine equations, so we look for an express that is easily factored that includes the higher degree terms (the terms hardest to work with):

$$(a - 2)(b - 2)(c - 2) = abc - 2(bc + ca + ab) + 4(a + b + c) - 8. \tag{2}$$

Now, we can use Equation 1 as a substitution equation in Equation 2:

$$(a - 2)(b - 2)(c - 2) = 8(a + b + c) - 8.$$

Or, more simply, we let $x = a - 2$, $y = b - 2$, and $z = c - 2$ to get

$$xyz = 8(x + y + z) + 40.$$

The symmetry of our equation means we can let $x \leq y \leq z$ without loss of generality. Now we can narrow our work down into cases by focusing on the smallest (most tightly bounded) variable. We have that

$$\begin{aligned}xyz &\geq xz^2,\\ 3x &\geq x + y + z.\end{aligned}$$

So, we can narrow our focus (essentially) to one variable:

$$24x + 40 \geq 8(x + y + z) + 40 = xyz \geq xz^2.$$

Thus, when $z \geq 6$, we have

$$24x + 40 \geq xz^2 \geq 36x \qquad \Rightarrow \qquad 40 \geq 12x,$$

but then $z \geq 6 > 4/3 \geq x$, a contradiction. So, we have narrowed our hunt for solutions to five cases:

- When $z = 1$, we have

  $$xy = 8(x + y) + 48 \qquad \Leftrightarrow \qquad (x - 8)(y - 8) = 112.$$

  Hunting down pairs of divisors of 112, we find solutions $(x, y) = (22, 16), (24, 15), (36, 12), (64, 10)$, and $(120, 9)$.
- When $z = 2$, we have

  $$2xy = 8(x + y) + 56 \qquad \Leftrightarrow \qquad (x - 4)(y - 4) = 44.$$

  Hunting down pairs of divisors of 44, we find solutions $(x, y) = (15, 8), (26, 6)$, and $(48, 5)$.
- When $z = 3$, we have

  $$3xy = 8(x + y) + 64 \qquad \Leftrightarrow \qquad y = \frac{8x + 64}{3x - 8}.$$

  Applying the Euclidean Algorithm for Polynomials, we find that

  $$\begin{aligned} \gcd(8x + 64, 3x - 8) &= \gcd(2x + 80, 3x - 8) \\ &= \gcd(2x + 80, x - 88) \\ &= \gcd(256, x - 88). \end{aligned}$$

  The fraction $\frac{8x+64}{3x-8}$ is an integer only if $3x - 8$ is that GCD, which is to say that $3x - 8$ must be a divisor of 256. We find solutions $(x, y) = (8, 8), (24, 4)$, and $(88, 3)$.
- When $z = 4$, we have

  $$4xy = 8(x + y) + 72 \qquad \Leftrightarrow \qquad (x - 2)(y - 2) = 22.$$

  Hunting down pairs of divisors of 22, we find solutions $(x, y) = (13, 4)$ (we discard $(x, y) = (23, 3)$ since in that case $z > y$).
- When $z = 5$, we have

  $$5xy = 8(x + y) + 80 \qquad \Leftrightarrow \qquad y = \frac{8x + 80}{5x - 8}.$$

  Applying the Euclidean Algorithm for Polynomials, we find that

  $$\begin{aligned} \gcd(8x + 80, 5x - 8) &= \gcd(3x + 88, 5x - 8) \\ &= \gcd(3x + 88, 2x - 96) \\ &= \gcd(x + 184, 2x - 96) \\ &= \gcd(x + 184, -464) = \gcd(x + 184, 464). \end{aligned}$$

  So, we look for solutions where $5x - 8$ is a divisor of 464. These are congruent to 2 modulo 5, which helps our hunt. All integers $x - 2$ with this property lead to either $x < z$ or $y < z$, so there are no solutions for this case.

Now we note the value of s for each solution:

$$\begin{array}{ccccccc}
\underline{(x,y,z)} & & \underline{(a,b,c)} & & & & \\
(22,16,1) & \to & (24,18,3) & \to & s & = & 24+18+3 = 45, \\
(24,15,1) & \to & (26,17,3) & \to & s & = & 26+17+3 = 46, \\
(36,12,1) & \to & (38,14,3) & \to & s & = & 38+14+3 = 55, \\
(64,10,1) & \to & (66,12,3) & \to & s & = & 66+12+3 = 81, \\
(120,9,1) & \to & (122,11,3) & \to & s & = & 122+11+3 = 136, \\
(15,8,2) & \to & (17,10,4) & \to & s & = & 17+10+4 = 31, \\
(26,6,2) & \to & (28,8,4) & \to & s & = & 28+8+4 = 40, \\
(48,5,2) & \to & (50,7,4) & \to & s & = & 50+7+4 = 61, \\
(8,8,3) & \to & (10,10,5) & \to & s & = & 10+10+5 = 25, \\
(24,4,3) & \to & (26,6,5) & \to & s & = & 26+6+5 = 37, \\
(88,3,3) & \to & (90,5,5) & \to & s & = & 90+5+5 = 100, \\
(13,4,4) & \to & (15,6,6) & \to & s & = & 15+6+6 = 27.
\end{array}$$

Now we sum these possible values of s (arranging them in increasing order to make sure we don't repeat any values):

$$25+27+31+37+40+45+46+55+61+81+100+136 = 684.$$

Round 4 Problems and Solutions

1. A fair 20-sided die has faces numbered 1 through 20. The die is rolled three times and the outcomes are recorded. If a and b are relatively prime integers such that a/b is the probability that the three recorded outcomes can be the sides of a triangle with positive area, find $a+b$.

   Answer: 1201

   Solution: We let p be the probability that the three recorded results, x, y, and z form the sides of a triangle with positive area. By the Triangle Inequality, p is the probability that each of the following inequalities is simultaneously satisfied:

$$\begin{aligned} x &< y+z, \\ y &< x+z, \\ z &< x+y. \end{aligned}$$

   Much simpler than finding the intersection of these inequalities is noting that if they are not all satisfied, then *exactly* one of the following inequalities is true:

$$\begin{aligned} x &\geq y+z, \\ y &\geq x+z, \\ z &\geq x+y. \end{aligned}$$

   These inequalities comprise cases that are symmetric. So, we can count the cases in which just the first is true, and multiply that result by 3.

   We count the possibilities such that $x \geq y+z$ using an organized list, in which each next row becomes apparent. Since $y, z \geq 1$, we know $x \geq 2$, so we start there:

$$\begin{array}{lcl} x=2 & \rightarrow & (y,z)=(1,1), \\ x=3 & \rightarrow & (y,z)=(1,1),(1,2),(2,1), \\ x=4 & \rightarrow & (y,z)=(1,1),(1,2),(1,3),(2,1),(2,2),(3,1), \\ & \vdots & \\ x=19 & \rightarrow & (y,z)=(1,1),\ldots,(18,1) \\ x=20 & \rightarrow & (y,z)=(1,1),\ldots,(19,1). \end{array}$$

   In each new row, $x=n$, we including all the ordered pairs (y,z) from the previous row, plus solutions to $y+z=n$, of which there are exactly $n-1$ in which y and z are positive integers. The total number of solutions where $x=n$ is thus

$$1+2+\cdots+(n-1)=\frac{n(n-1)}{2}=\frac{1}{2}n^2-\frac{1}{2}n.$$

   Thus, the overall number of ordered triples we are looking for is

$$\begin{aligned} \sum_{n=1}^{20}\left(\frac{1}{2}n^2-\frac{1}{2}n\right) &= \frac{1}{2}\sum_{n=1}^{20}n^2-\frac{1}{2}\sum_{n=1}^{20}n \\ &= \frac{1}{2}\left(\frac{20\cdot 21\cdot 41}{6}\right)-\frac{1}{2}\left(\frac{20\cdot 21}{2}\right) \\ &= 1435-105=1330. \end{aligned}$$

The total number of ordered triplets (x, y, z) of results that do not form a triangle of positive area is $3 \cdot 1330$, so the total number that do is $20^3 - 3 \cdot 1330 = 4010$. The probability is thus

$$\frac{4010}{20^3} = \frac{4010}{8000} = \frac{401}{800}.$$

Hence, the answer is $401 + 800 = 1201$.

2. Let m be the maximum possible value of $x^{16} + \frac{1}{x^{16}}$, where

$$x^6 - 4x^4 - 6x^3 - 4x^2 + 1 = 0.$$

Find the remainder when m is divided by 2007.

Answer: 1865

Solution: Let $f(x) = x^6 - 4x^4 - 6x^3 - 4x^2 + 1$. Then the roots of $f(x)$ are the same as the roots of $f(1/x)$. This symmetry of reciprocal roots allows us to manipulate the equation into a form much like the expression we intend to compute. Dividing the given equation by x^3, we have

$$x^3 - 4x - 6 - \frac{4}{x} + \frac{1}{x^3} = 0. \tag{3}$$

Now, letting $y = x + \frac{1}{x}$, we note that

$$y^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}.$$

So, we can rewrite Equation 3 as a polynomial equation in y:

$$y^3 - 7y - 6 = 0.$$

Factoring, we find that $y^3 - 7y - 6 = (y+1)(y+2)(y-3) = 0$, so $y = -1$, $y = -2$, or $y = 3$.

Now we know that one of the following is true:

$$\begin{aligned} x + \frac{1}{x} &= -1, & (4) \\ x + \frac{1}{x} &= -2, & (5) \\ x + \frac{1}{x} &= 3. & (6) \end{aligned}$$

Following the computations that carry us from the last of these possibilities to our answer, we see that the need to carry out the other two sets of computations is unnecessary:

$$\begin{aligned} x + \frac{1}{x} = 3 \quad &\Rightarrow \quad x^2 + \frac{1}{x^2} = 3^2 - 2 = 7, \\ &\Rightarrow \quad x^4 + \frac{1}{x^4} = 7^2 - 2 = 47, \\ &\Rightarrow \quad x^8 + \frac{1}{x^8} = 47^2 - 2 = 2207, \\ &\Rightarrow \quad x^{16} + \frac{1}{x^{16}} = 2207^2 - 2. \end{aligned}$$

So, $m = 2207^2 - 2 \equiv 200^2 - 2 \equiv 39998 \equiv 1865 \pmod{2007}$.

3. A sequence $a_1, a_2, a_3, \dots$ is defined as follows: $a_1 = 2007$, and $a_n = a_{n-1} + n \pmod{k}$, where $0 \le a_n < k$. For how many values of k, where $2007 < k < 10^{12}$, does the sequence assume all k possible values (modulo k residues)?

Answer: 29

Credit: This problem is based on a similar problem from the 1991 APMO.

Solution: We begin by noting that the given sequence takes on all modulo k residues if and only if the following sequence assumes all modulo k residues: $b_1 = 1$, and $b_n = b_{n-1} + n \pmod{k}$, where $0 \le b_n < k$. This is because $b_n \equiv a_n - 2006 \pmod{k}$, where $a_n - 2006$ ranges through all modulo k residues. This simplifies matters since

$$b_n \equiv 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \pmod{k}.$$

Now we have a closed form to work with. It's easy to see that this closed form is periodic modulo k:

$$b_{n+k} - b_n = \frac{(n+k)(n+k+1)}{2} - \frac{n(n+1)}{2} = nk + \frac{k(k+1)}{2} \equiv \frac{k(k+1)}{2} \pmod{k}.$$

When k is odd, $k(k+1)/2 = km \equiv 0 \pmod{k}$, where $m = (k+1)/2$ is an integer. Thus, when k is odd, the sequence has a period of k. Otherwise, k is even, and we can show similarly that $2k$ is a period.

In addition to periodicity, we find that the sequence exhibits symmetry. When k is odd,

$$b_{k-n-1} = \frac{(k-n)(k-n-1)}{2} = k\left(\frac{k-1}{2} - n\right) + \frac{n(n+1)}{2} \equiv \frac{n(n+1)}{2} \equiv b_n \pmod{k}.$$

The period of k implies that the sequence ranges through all modulo k residues if and only if $b_1, b_2, \dots, b_k$ are distinct, but they are not. So we are only looking for even values of k.

When k is even, symmetry is still present within the period, but the period is $2k$:

$$b_{2k-n-1} = \frac{(2k-n-1)(2k-n)}{2} = \frac{(n+1-2k)(n-2k)}{2} \equiv \frac{n(n+1)}{2} \pmod{k}.$$

So, the sequence assumes all k possible values if and only if the first k terms assume all k possible values.

Plugging in possible even integers k, we find that $k = 2$, $k = 4$, and $k = 8$ work, while $k = 6$ and $k = 10$ do not. If we stop here and conjecture that powers of 2 work, we find further that $k = 12$ and $k = 14$ do not work, while $k = 16$ does.

First, we go about trying to prove that powers of 2 work. Supposing k is a power of 2, we note that $a_r = a_s$ for $1 \le r < s \le k$ iff

$$\frac{s(s+1)}{2} - \frac{r(r+1)}{2} = \frac{(s-r)(r+s+1)}{2} \equiv 0 \pmod{k} \qquad \Rightarrow \qquad (s-r)(r+s+1) \equiv 0 \pmod{2k}.$$

Both $s - r$ and $r + s + 1$ are less than $2k$, which is a power of 2. Thus, both $s - r$ and $r + s + 1$ are even. However, the parities of $s - r$ and $r + s + 1$ are different. Thus, it cannot be true that $a_r = a_s$, hence k can be any power of 2.

Now, note that if $p \mid k$ for some prime p, then

$$\frac{n(n+1)}{2} \equiv a \pmod{k} \qquad \Rightarrow \qquad \frac{n(n+1)}{2} \equiv a \pmod{p}.$$

So if k works, so does p. However, no odd numbers work, so $p = 2$ is the *only* prime divisor of k.

Finally, we tally the powers of 2 between 2007 and 10^{12}. First, we note that $2^{10} = 1024 < 2007 < 2048 = 2^{11}$, so 2^{11} is the smallest possible value of k. Next, we note that $2^{10} = 1024$, which is very close to 10^3. A bit of estimation and we find that $2^{39} < 10^{12} < 2^{40}$, so 2^{39} is the largest possible value of k. The number of possible values of k is thus $39 - 11 + 1 = 29$.

4. Bobby Fisherman played a tournament in which he played 2009 players. He either won or lost every game. He lost his first two games, but won 2002 total games. At the conclusion of each game, he computed his exact winning percentage at that moment. Let $w_1, w_2, \ldots, w_{2009}$ be his winning percentages after games 1, 2, ..., 2009 respectively. There are some real numbers, such as 0, which are necessarily members of the set $W = \{w_1, w_2, \ldots, w_{2009}\}$. How many positive numbers are in the intersection of all possible sets W?

   Answer: 286

   Credit: This problem was inspired by 2004 Putnam problem A-1.

   Solution: Given that we are dealing with discrete quantities (wins and losses), percentages are not the most enlightening form of the numbers we want to count. So, we imagine the possible win/loss ratios Fisherman could have during his tournament (after game 2). We write these win/loss ratios first as unreduced fractions, then as decimals to make a point:

$$\begin{array}{cccccccccc}
\frac{0}{2} & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots & \frac{998}{2} & \frac{999}{2} & \frac{1000}{2} & \cdots & \frac{2002}{2} \\
\frac{0}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots & \frac{998}{3} & \frac{999}{3} & \frac{1000}{3} & \cdots & \frac{2002}{3} \\
\frac{0}{4} & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots & \frac{998}{4} & \frac{999}{4} & \frac{1000}{4} & \cdots & \frac{2002}{4} \\
\frac{0}{5} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \cdots & \frac{998}{5} & \frac{999}{5} & \frac{1000}{5} & \cdots & \frac{2002}{5} \\
\frac{0}{6} & \frac{1}{6} & \frac{2}{6} & \frac{3}{6} & \frac{4}{6} & \cdots & \frac{998}{6} & \frac{999}{6} & \frac{1000}{6} & \cdots & \frac{2002}{6} \\
\frac{0}{7} & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \cdots & \frac{998}{7} & \frac{999}{7} & \frac{1000}{7} & \cdots & \frac{2002}{7}
\end{array}$$

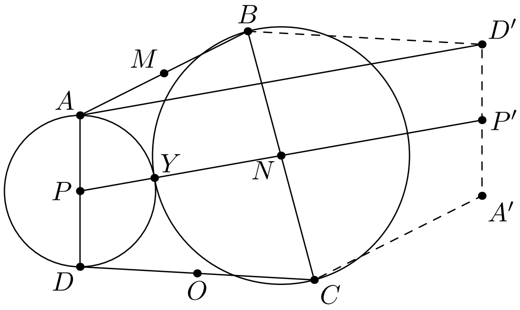
   Fisherman's W/L ratios "travel" from the upper-left-hand side of this grid to the lower-right-hand side by moving right or down one fraction at a time. While some of the numbers that must be in set W may already be obvious, it's finding numbers that aren't necessarily in W that can be just as enlightening. The very last row makes many of these values most obvious. The decimal representations of fractions with denominator 7 can only be matched in other rows when they are integers.

   From the set $W = \{w_1, w_2, \ldots, w_{2009}\}$, we construct a set $X = \{x_1, x_2, \ldots, x_{2009}\}$ of the analogous W/L ratios. Given the one-to-one correspondence of values in X and W, we can rephrase the problem in terms of finding rational numbers that must be elements of X. Now, thinking of the elements of X in terms of decimals, we reconstruct the table above with the fractions replaced by their decimal equivalents (rounded to the nearest hundredth):

0	0.5	1	1.5	2	$\cdots$	499	499.5	500	$\cdots$	1001
0	0.33	0.67	1	1.33	$\cdots$	322.67	333	333.33	$\cdots$	667.33
0	0.25	0.5	0.75	1	$\cdots$	249.5	249.75	250	$\cdots$	500.5
0	.0.2	0.4	0.6	0.8	$\cdots$	199.6	199.8	200	$\cdots$	400.4
0	0.17	0.33	0.5	0.67	$\cdots$	166.33	166.5	166.67	$\cdots$	333.67
0	0.14	0.28	0.43	0.57	$\cdots$	142.57	142.71	142.86	$\cdots$	286

Now, following a path from the upper-left-hand corner to the lower-right-hand corner, we can go straight down to the last row, and the only numbers that might be found in other rows are the integers. In fact, we see that Fisherman's W/L ratio has a low of 0, a possible maximum of 1001, and winds up at 286 at the end of all 2009 games. The positive integers from 1 to 286 are the only positive numbers that are candidates to be found in all possible sets X.

Now, in order to see that all the integers from 1 to 286 inclusive *must* be found in X, we think about the graph of the points (n, x_n), where $2 \leq n \leq 2009$. This graph usually rises (moves right on the decimal table above), but sometimes sinks on any given "step" (movement of one unit to the right on the graph). But as n increases, no integer (in the range 1 to 286 inclusive) is ever skipped as a possible value of x_n because each of those integers is in every row in the table above. Thus, there are exactly 286 integers which must be elements of X, corresponding to 286 winning percentages which must be elements of W.



5. Convex quadrilateral $ABCD$ has the property that the circles with diameters AB and CD are tangent at point X inside the quadrilateral, and likewise, the circles with diameters BC and DA are tangent at a point Y inside the quadrilateral. Given that the perimeter of $ABCD$ is 96, and the maximum possible length of XY is m, find $\lfloor 2007m \rfloor$.

   Answer: 0

   Credit: Problem and solution by Harvard University undergraduate Zachary Abel.

   Solution: Let M, N, O, P be the midpoints of AB, BC, CD, DA as illustrated. Rotate the quadrilateral 180° around N. By triangle inequality we have $AB + BD' \geq AD'$, or equivalently,

$$AB + CD \geq PP' = 2PN.$$

   Equality holds if and only if A, B, and D' are collinear, i.e. AB and CD are parallel. Similarly,

$$BC + DA \geq 2MO,$$

   with equality holding if and only if BC is parallel to DA.

   But equality must hold in both cases: indeed, by the circle tangency we find $PN = PY + YN = \frac{1}{2}(AD + BC)$, and likewise $MO = \frac{1}{2}(AB + CD)$, and adding these shows that

$$AB + BC + CD + DA = 2MO + 2PN.$$

   Thus $AB \parallel CD$ and $BC \parallel DA$, i.e. $ABCD$ is a parallelogram. Furthermore, since $BC = MO = \frac{1}{2}(AB + CD) = AB$, the quadrilateral is in fact a rhombus. Finally, if Z is the center of the rhombus then the circles centered at M and O pass through Z (because $\angle AZB = \angle CZD = 90°$), so $Z = X$ is the point of tangency. Likewise, $Z = Y$. So $XY = 0$, and $\lfloor 2007 \cdot 0 \rfloor = 0$.

Round 5 Problems and Solutions

1. Find the smallest positive integer n such that a cube with sides of length n can be divided up into exactly 2007 smaller cubes, each of whose sides is of integer length.

   Answer: 13

   Credit: Inspired by a problem from the 1996 Iberoamerican Mathematics Olympiad.

   Solution: We note that $12^3 = 1728 < 2007 < 2197 = 13^3$, making 13 the smallest possible candidate for n. The trick is figuring out whether or not we can construct such a cube from exactly 2007 smaller cubes with integer side lengths.

   Since $5^3 < 190 < 6^3$, none of the smaller cubes can have side length greater than 5. Letting a, b, c, d, and e be the number of smaller cubes of sides of length 1, 2, 3, 4, and 5, we have the system of equations

   $$\begin{aligned} a+b+c+d+e &= 2007, \\ a+8b+27c+64d+125e &= 2197. \end{aligned}$$

   Subtracting the first equation from the second, we get

   $$7b+26c+63d+124e=190.$$

   Now we just hunt down a solution, such as $(a,b,c,d,e)=(2002,2,2,0,1)$. These 2007 smaller cubes are so small, with the smallest (unit cubes) being so numerous that it's clear that we can pack them into a $13 \times 13 \times 13$ cube.

2. In the game of *Winners Make Zeros*, a pair of positive integers (m,n) is written on a sheet of paper. Then the game begins, as the players make the following legal moves:

   - If $m \geq n$, the player chooses a positive integer c such that $m-cn \geq 0$, and replaces (m,n) with $(m-cn,n)$.
   - If $m < n$, the player chooses a positive integer c such that $n-cm \geq 0$, and replaces (m,n) with $(m,n-cm)$.

   When m or n becomes 0, the game ends, and the last player to have moved is declared the winner. If m and n are originally 2007777 and 2007, find the largest choice the first player can make for c (on his first move) such that the first player has a winning strategy after that first move.

   Answer: 999

   Credit: This problem is base on the game usually known as *Euclid*.

   Solution: First, we note that the game jumps through "steps" in the Euclidean Algorithm.

   Now, note that from any position, a player cannot have multiple moves that lead to winning positions. If $c=c_1$ and $c=c_2$ lead to winning positions, where $c_1 > c_2$, then after choosing c_2, a players opponent could let $c=c_1-c_2$ to take a winning position, contradicting the fact that both $c=c_1$ and $c=c_2$ both lead to winning positions. So, we are looking for the one value of c that the first player can use to leave himself with a winning strategy.

   Next, since the game is necessarily finite, we can construct all possible sets of moves. This means that a player either has a winning strategy, or they are in a losing position (so long as their opponent plays perfectly). However, if a player has more than one move, they cannot be in a losing position, because they could always choose $c = 1$, leaving their opponent with a subset of their available moves. This means that a player with more than one legal move necessarily has a winning position.

Now, let's take a look at parts of the Euclidean Algorithm that take us from the beginning of the game to the end, noting each point at which a player's move would be forced (because $c = 1$ leads to the only legal move):

(m, n)	Next Move Forced
$(2007777, 2007)$	
$(2005770, 2007)$	
$\vdots$	
$(4791, 2007)$	
$(2784, 2007)$	✓
$(777, 2007)$	
$(777, 1230)$	✓
$(777, 453)$	✓
$(324, 453)$	✓
$(324, 129)$	
$(195, 129)$	✓
$(66, 129)$	✓
$(66, 63)$	✓
$(3, 63)$	

The player who gets to move when $(m, n) = (3, 63)$ wins by replacing $(3, 63)$ with $(3, 0)$. Walking backward through the moves, the player with the winning strategy wants to avoid having their turn when $(m, n) = (66, 63)$. This means they want to have their turn when $(m, n) = (66, 129)$. They want to avoid $(m, n) = (195, 129)$. They want their turn when $(m, n) = (324, 129)$. They want to avoid $(m, n) = (324, 453)$. They want their turn when $(m, n) = (777, 453)$. They want to avoid $(m, n) = (777, 1230)$. They want their turn when $(m, n) = (777, 2007)$. So, $(m, n) = (2784, 2007)$ is a losing position, where

$$2784 = 2007777 - c \cdot 2007 \qquad \Rightarrow \qquad c = 999.$$

3. (5 points) Find the sum of all integers n such that

$$n^4 + n^3 + n^2 + n + 1.$$

is a perfect square.

Answer: 2

Solution: Looking for ways to approximate the square root of $n^4 + n^3 + n^2 + n + 1$, we note that

$$\begin{aligned}
\left(n^2 + \frac{1}{2}n\right)^2 &= n^4 + n^3 + \frac{1}{4}n^2, \\
\left(n^2 + \frac{1}{2}n + \frac{1}{2}\right)^2 &= n^4 + n^3 + \frac{5}{4}n^2 + \frac{1}{2}n + \frac{1}{4}, \\
\left(n^2 + \frac{1}{2}n + 1\right)^2 &= n^4 + n^3 + \frac{9}{4}n^2 + n + 1.
\end{aligned}$$

We note that

$$(n^4+n^3+n^2+n+1)-\left(n^4+n^3+\frac{1}{4}n^2\right)=\frac{3}{4}n^2+n+1>0 \tag{7}$$

for all real numbers n, and

$$\left(n^4+n^3+\frac{9}{4}n^2+n+1\right)-(n^4+n^3+n^2+n+1)=\frac{5}{4}n^2\geq 0 \tag{8}$$

for all real numbers n, with equality iff $n=0$. In summary,

$$\left(n^2+\frac{1}{2}n\right)^2<n^4+n^3+n^2+n+1\leq\left(n^2+\frac{1}{2}n+1\right)^2.$$

So, since either $n^2+\frac{1}{2}n+\frac{1}{2}$ or $n^2+\frac{1}{2}n+1$ is an integer depending on the parity of n, we know that when $n^4+n^3+n^2+n+1$ is a perfect square, it is equal to the square of one of those quantities.

If equality occurs in Inequality 8, then $n=0$, in which case we have $n^4+n^3+n^2+n+1=1^2$. Otherwise, we have that

$$\left(n^2+\frac{1}{2}n+\frac{1}{2}\right)^2=n^4+n^3+\frac{5}{4}n^2+\frac{1}{2}n+\frac{1}{4}=n^4+n^3+n^2+n+1,$$

so

$$\frac{1}{4}n^2-\frac{1}{2}n-\frac{3}{4}=0 \qquad\Leftrightarrow\qquad n^2-2n-3=(n+1)(n-3)=0.$$

Thus, either $n=-1$, in which case $n^4+n^3+n^2+n+1=1^2$, or else $n=3$, in which case $n^4+n^3+n^2+n+1=11^2$. Hence, our answer is $0+(-1)+3=2$.

4. In triangle ABC, points A', B', and C' are chosen with A' on segment AB, B' on segment BC, and C' on segment CA so that triangle $A'B'C'$ is directly similar to ABC. The incenters of ABC and $A'B'C'$ are I and I' respectively. Lines BC, $A'C'$, and II' are all parallel. If $AB=100$ and $AC=120$, what is the length of BC?

   Answer: 144

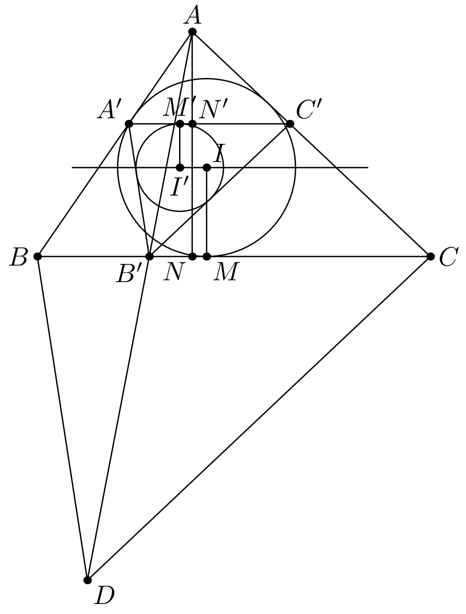
   Credit: Problem and solution by Harvard University undergraduate Zachary Abel.

   Solution: We'll use the standard notation $BC=a$, $CA=b$, $AB=c$. Choose the point D so that $ABC\sim BDC$ as shown in the diagram; the ratio between the triangles is $BC/AC=a/b$. Triangles $A'B'C'$ and BDC are homothetic with center A, so B' is the intersection of AD with BC. Since CB' bisects angle ACD, the angle bisector theorem gives

$$\frac{AB'}{AD}=\frac{AC}{AC+CD}=\frac{b}{b+\frac{a^2}{b}}=\frac{b^2}{a^2+b^2}.$$

   So $A'B'C'$ and BDC are similar with this ratio, i.e. $A'B'C'$ and ABC have similarity ratio $\frac{a}{b}\cdot\frac{b^2}{a^2+b^2}=\frac{ab}{a^2+b^2}$.

   Now let $r=IM$ be the inradius of ABC, and h the length of altitude AN from A to BC. Likewise set $r'=I'M'=\frac{ab}{a^2+b^2}r$ as the inradius of $A'B'C'$ and $h'=AN'=\frac{b^2}{a^2+b^2}h$ as the length of the altitude to



$A'C'$. The perpendicular distance from A to BC is $AN = h$, but it is also $AN' + M'I' + IM = h' + r + r'$. So $r + r' = h - h'$, i.e., by substituting the above values, $(a^2 + ab + b^2)r = a^2 h$.

Finally, the area of ABC can be expressed as $\frac{1}{2}ah = \frac{1}{2}(a + b + c)r$, and this combined with the above equation gives $(a^2 + ab + b^2)r = (a^2 + ab + ac)r$, i.e. $b^2 = ac$. Since $b = 120$ and $c = 100$, the answer is $a = b^2/c = 144$.

5. Let c be the number of ways to choose three vertices of an 6-dimensional cube that form an equilateral triangle. Find the remainder when c is divided by 2007.

   Answer: 233

   Credit: The more general problem of finding the number of ways to select vertices from an n-dimension cube that form an equilateral triangle was proposed by Neven Juric, and a solution appeared in the November 1995 volume of the *The College Mathematics Journal* published by the Mathematical Association of America.

   Solution: We could jump straight into the specific problem. We choose instead to begin with a discussion of the problem for an n-dimensional cube. First, we get a handle on the problem by assigning coordinates to the vertices. The coordinates of each vertex are an n-tuple of 0's and 1's. Now, assume one of the three vertices that make up some equilateral triangle are at the origin, which we label O, and let the other two vertices of the triangle be A and B. When m is the number of 1's in the coordinates of A, then the sides of the equilateral triangle are of length $\sqrt{m}$. We can permute the dimensions to relate one case for the coordinates of A to all other cases, so let us examine the simple(st) case where

$$A = (\underbrace{1, 1, \ldots, 1}_{m}, \underbrace{0, 0, \ldots, 0}_{n-m}).$$

Since $OB = \sqrt{m}$, the coordinates of B also include exactly m 1's, and since $AB = \sqrt{m}$, exactly half the 1's in each A and B agree with one another. Thus, m is even, and we let $m = 2k$, and we have equilateral triangles with coordinates such as

$$
\begin{aligned}
O &= (0,\ldots, \qquad\qquad \ldots, \qquad\qquad \ldots, \qquad\qquad \ldots,0),\\
A &= (\underbrace{1,1,\ldots,1}_{k},\ \underbrace{1,1,\ldots,1}_{k},\ \underbrace{0,0,\ldots,0}_{k},\ \underbrace{0,0,\ldots,0}_{n-3k}),\\
B &= (\underbrace{1,1,\ldots,1}_{k},\ \underbrace{0,0,\ldots,0}_{k},\ \underbrace{1,1,\ldots,1}_{k},\ \underbrace{0,0,\ldots,0}_{n-3k}).
\end{aligned}
$$

The value of k is at most $n/3$. So, permuting the n dimensions (columns of coordinates above), we find that there are

$$\sum_{k=1}^{\lfloor n/3 \rfloor} \frac{n!}{(n-3k)!k!k!k!}$$

equilateral triangles that include O. Now, we could have selected any one of the 2^n vertices as O, and we would be overcounting by a factor of 3!, so the total number of equilateral triangles is

$$\frac{2^n}{6} \sum_{k=1}^{\lfloor n/3 \rfloor} \frac{n!}{(n-3k)!k!k!k!}.$$

In particular, when $n = 6$, we have that

$$
\begin{aligned}
\frac{2^6}{6} \sum_{k=1}^{2} \frac{6!}{(6-3k)!k!k!k!} &= \frac{32}{3}\left(\frac{6!}{3!1!1!1!} + \frac{6!}{0!2!2!2!}\right)\\
&= \frac{32}{3}(120+90) = 2240.
\end{aligned}
$$

So, $c = 2240 \equiv 233 \pmod{2007}$, and 233 is our answer.

Round 6 Problems and Solutions

1. Given that

$$\begin{aligned} x &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2007}, \\ y &= \frac{1}{1005} + \frac{1}{1006} + \frac{1}{1007} + \cdots + \frac{1}{2007}, \end{aligned}$$

find the value of k such that

$$x = y + \frac{1}{k}.$$

Answer: 1004

Solution: Familiarity with Catalan's Identity might lead a student toward a quick solution:

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Inserting a couple of fractions in the series that represent x and y above gives us Catalan's Identity for $n = 1004$:

$$x - \frac{1}{2008} = y + \frac{1}{2008} \qquad \Leftrightarrow \qquad x = y + \frac{1}{1004}.$$

While many problem solvers may both recall and recognize Catalan's Identity quickly, clever problem solvers can solve the given problem in a manner that generalizes as a proof of Catalan's Identity. We begin by looking for a series that we can relate simultaneously to both x and y. Some obervers might already note that subtracting $1/2008$ from x and adding the same fraction to y creates two series in which the first has exactly twice as many terms as the second, with equally many terms added and subtracted.

We compare x to y by separating the series for x into a series in which reciprocals of consecutive integers are added, and another series:

$$x = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2007} = \sum_{m=1}^{2007} \frac{1}{m} - 2\sum_{m=1}^{1003} \frac{1}{2m}.$$

Now we simplify the second summation and cancel terms:

$$\sum_{m=1}^{2007} \frac{1}{m} - 2\sum_{m=1}^{1003} \frac{1}{2m} = \sum_{m=1}^{2007} \frac{1}{m} - \sum_{m=1}^{1003} \frac{1}{m} = \sum_{m=1004}^{2007} \frac{1}{m} = \frac{1}{1004} + y.$$

So, $x = y + 1/1004$, making $k = 1004$.

2. Let

$$S = 1 + \frac{1}{8} + \frac{1\cdot 5}{8\cdot 16} + \frac{1\cdot 5\cdot 9}{8\cdot 16\cdot 24} + \cdots + \frac{1\cdot 5\cdot 9\cdots(4k+1)}{8\cdot 16\cdot 24\cdots(8k+8)} + \cdots.$$

Find the positive integer n such that $2^n < S^{2007} < 2^{n+1}$.

Answer: 501

Solution: There seems to be no simple way to evaluate the series based on such simple series as arithmetic or geometric series. So, we examine each term, hoping to "repackage" them in a way that makes the value of S more apparent:

$$\begin{aligned}
\frac{1\cdot 5\cdot 9\cdots(4k-3)}{8\cdot 16\cdot 24\cdots 8k} &= \frac{1}{2^k}\frac{1\cdot 5\cdot 9\cdots(4k-3)}{4\cdot 8\cdot 12\cdots 4k} \\
&= \frac{1}{2^k\cdot k!}\cdot\frac{1}{4}\cdot\frac{5}{4}\cdot\frac{9}{4}\cdots\frac{4k-3}{4} \\
&= \frac{1}{(-2)^k\cdot k!}\cdot\left(-\frac{1}{4}\right)\cdot\left(-\frac{5}{4}\right)\cdot\left(-\frac{9}{4}\right)\cdots\left(-\frac{4k-3}{4}\right) \\
&= \left(\frac{1}{2}\right)^k\binom{-1/4}{k}.
\end{aligned}$$

The terms in the given series correspond to those in a particular binomial expansion:

$$S = \left(1-\frac{1}{2}\right)^{-1/4} = \sqrt[4]{2}.$$

So, we note that

$$2^n < S^{2007} = 2^{501.75} < 2^{n+1},$$

hence $n = 501$.

3. Find the real number k such that a, b, c, and d are real numbers that satisfy the system of equations

$$\begin{aligned}
abcd &= 2007, \\
a &= \sqrt{55+\sqrt{k+a}}, \\
b &= \sqrt{55-\sqrt{k+b}}, \\
c &= \sqrt{55+\sqrt{k-c}}, \\
d &= \sqrt{55-\sqrt{k-d}}.
\end{aligned}$$

Answer: 1018

Solution: We begin by examining the four similar equations. Squaring each, manipulating, and squaring again, these equations become

$$\begin{aligned}
(a^2-55)^2 &= k+a, \\
(b^2-55)^2 &= k+b, \\
(c^2-55)^2 &= k-c, \\
(d^2-55)^2 &= k-d.
\end{aligned}$$

These equations are extremely similar, and hint at some polynomial whose roots relate to a, b, c, and d. Looking for a way to induce sameness from similarity, we note that

$$\begin{aligned}(a^2-55)^2 &= k+a,\\ (b^2-55)^2 &= k+b,\\ ((-c)^2-55)^2 &= k+(-c),\\ ((-d)^2-55)^2 &= k+(-d).\end{aligned}$$

Thus, a, b, $-c$, and $-d$ are the roots of

$$(x^2-55)^2 = k+x \qquad \Leftrightarrow \qquad x^4-110x^2-x+3025-k=0.$$

By Vieta, the product of these roots is $ab(-c)(-d) = 3025-k$, so $abcd = 3025-k$. But $abcd = 2007$, so $2007 = 3025-k$. Hence, $k = 1018$.

4. Let $x_1, x_2, \ldots, x_{2007}$ be real numbers such that $-1 \le x_i \le 1$ for $1 \le i \le 2007$, and

$$\sum_{i=1}^{2007} x_i^3 = 0.$$

Find the maximum possible value of $\sum_{i=1}^{2007} x_i$.

Answer: 669

Solution: We consider a cubic polynomial $p(x) = x^3+ax^2+bx+c$ such that

$$\sum_{i=1}^{2007} p(x_i) = \sum_{i=1}^{2007} x_i^3 + a\sum_{i=1}^{2007} x_i^2 + b\sum_{i=1}^{2007} x_i + nc \overset{\geq}{\leq} 0$$

gives us a comparison between $\sum_{i=1}^{2007} x_i$ and 0 (and we don't yet know which direction we would like the inequality sign to face). We want the roots of p to be in the interval $[-1,1]$, so that we can achieve equality, where our maximum will occur. In order to get rid of the sum of squares of the variables, we also want the sum of the roots to be 0.

A couple of possibilities seem easiest to construct:

$$\begin{aligned}(x-1)\left(x+\frac{1}{2}\right)^2 &\leq 0,\\ (x+1)\left(x-\frac{1}{2}\right)^2 &\geq 0.\end{aligned}$$

As an exercise, see if you can find other polynomials that would help solve this problem. The first stems from the fact that $x_i - 1 \le 0$, and the Trivial Inequality. The second stems from the fact that $x_i + 1 \ge 0$, along with the Trivial Inequality. Expanding the LHS of each, we get

$$\begin{aligned} x^3 - \frac{3}{4}x - \frac{1}{4} &\le 0, \\ x^3 - \frac{3}{4}x + \frac{1}{4} &\ge 0. \end{aligned}$$

Summing where $x = x_i$, we have

$$\begin{aligned} \sum_{i=1}^{2007} x_i^3 - \frac{3}{4}\sum_{i=1}^{2007} x_i - \frac{2007}{4} &\le 0, \\ \sum_{i=1}^{2007} x_i^3 - \frac{3}{4}\sum_{i=1}^{2007} x_i + \frac{2007}{4} &\ge 0. \end{aligned}$$

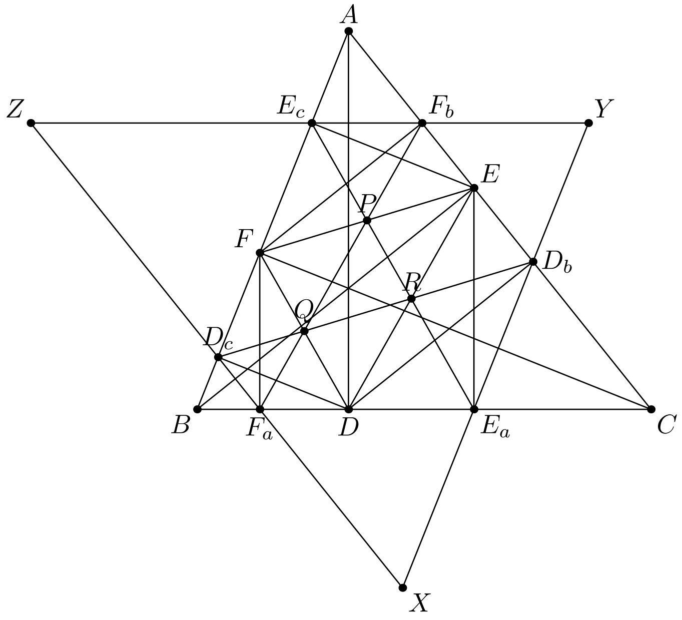
Since the sum of the cubes of the x_i is just 0, we have two inequalities that give bounds for the sum of the x_i themselves:

$$\begin{aligned} -\frac{3}{4}\sum_{i=1}^{2007} x_i - \frac{2007}{4} \le 0 \qquad &\Leftrightarrow \qquad \sum_{i=1}^{2007} x_i \ge -669, \\ -\frac{3}{4}\sum_{i=1}^{2007} x_i + \frac{2007}{4} \ge 0 \qquad &\Leftrightarrow \qquad \sum_{i=1}^{2007} x_i \le 669. \end{aligned}$$

We achieve the maximum of 669 with 8 of the x_i equal to $1/2$ for every 1 of them equal to -1:

$$\begin{aligned} 8 \cdot \left(\frac{1}{2}\right)^3 + 1 \cdot (-1)^3 &= 0, \\ 8 \cdot \left(\frac{1}{2}\right) + 1 \cdot (-1) &= 3, \end{aligned}$$

where $223 \cdot 9 = 2007$ and $223 \cdot 3 = 669$.



5. Acute triangle ABC has altitudes AD, BE, and CF. Point D is projected onto AB and AC to points D_c and D_b respectively. Likewise, E is projected to E_a on BC and E_c on AB, and F is projected to F_a on BC and F_b on AC. Lines D_bD_c, E_cE_a, F_aF_b bound a triangle of area T_1, and lines E_cF_b, D_bE_a, F_aD_c bound a triangle of area T_2. What is the smallest possible value of the ratio T_2/T_1?

   Answer: 25

   Credit: Problem and solution by Harvard University undergraduate Zachary Abel.

   Solution: We'll use the common notations $BC = a$, $CA = b$, $AB = c$, $\angle BAC = \alpha$, $\angle CBA = \beta$, $\angle ACB = \gamma$. Let the two described triangles be PQR and XYZ as in diagram 5. First, we can calculate

$$\angle D_cD_bD = \angle D_cAD = 90 - \beta = 90 - \angle DEC = \angle D_bDE.$$

   Since triangle ED_bD is right, this is enough to conclude that D_bD_c passes through the midpoint of DE, and for similar reasons, it bisects DF. As similar statements hold for E_cE_a and F_bF_c, triangle PQR must be the medial triangle of DEF. More angle chasing reveals that $\angle AE_cF_b = \angle AEF = \angle ABC$, so $YZ \parallel BC$, and likewise for XY and XZ. So triangles ABC and XYZ are similar.

   First we calculate the area $[PQR] = \frac{1}{4}[DEF]$. Since $AE = AB\cos\alpha = c\cos\alpha$ and $AF = b\cos\alpha$, we have $[AEF]/[ABC] = \cos^2\alpha$, and likewise around. So

$$\begin{aligned}\frac{[PQR]}{[ABC]} &= \frac{[DEF]}{4[ABC]}\\ &= \frac{1}{4}\left(1 - \frac{[AEF]}{[ABC]} - \frac{[BFD]}{[ABC]} - \frac{[CDE]}{[ABC]}\right)\\ &= \frac{1}{4}\left(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma\right).\end{aligned}$$

   Designate $\cos^2\alpha - \cos^2\beta - \cos^2\gamma = \rho$, so this area ratio is just $\frac{1}{4}(1-\rho)$.

Next we look at $[XYZ]$. We have $AE_c = AE\cos\alpha = AB\cos^2\alpha = c\cos^2\alpha$, and since $AE_cF_b \sim ABC$, $E_cF_b = a\cos^2\alpha$. We can then calculate $D_cE_c = c - AE_c - BD_c = c(1-\cos^2\alpha-\cos^2\beta)$, and since $D_cE_cZ \sim ABC$, we have $ZE_c = a(1-\cos^2\alpha-\cos^2\gamma)$. Similarly, $F_bY = a(1-\cos^2\alpha-\cos^2\gamma)$, so $YZ = YF_b + F_bE_c + E_cZ = a(2-\rho)$. This means $[XYZ]/[ABC] = (2-\rho)^2$.

Thus, the desired area ratio is

$$\frac{[XYZ]}{[PQR]} = \frac{T_2}{T_1} = \frac{4(2-\rho)^2}{1-\rho}.$$

On the interval $[\frac{3}{4}, 1)$ this is an increasing function of ρ, so proving the inequality $\frac{3}{4} \le \rho < 1$ would suffice to show that T_2/T_1 is minimized at

$$\frac{4(2-\frac{3}{4})^2}{1-\frac{3}{4}} = 25.$$

The inequality

$$\frac{3}{4} \le \cos^2\alpha + \cos^2\beta + \cos^2\gamma < 1 \tag{9}$$

can be shown in many ways. We'll utilize the identity

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 - 2\cos\alpha\cos\beta\cos\gamma,$$

which is readily verified. The right hand side of (9) follows directly from this identity because $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are positive for ABC acute. The other inequality is equivalent to $\cos\alpha\cos\beta\cos\gamma \le \frac{1}{8}$. Using AM-GM on BD and CD we find $a = BD + CD \ge 2\sqrt{BD \cdot CD} = 2\sqrt{bc\cos\beta\cos\gamma}$, and multiplying this with the other two similar inequalities $b \ge 2\sqrt{ca\cos\gamma\cos\alpha}$ and $c \ge 2\sqrt{ab\cos\alpha\cos\beta}$ indeed gives $\frac{1}{8} \ge \cos\alpha\cos\beta\cos\gamma$, as needed. Equality holds if (and only if) ABC is equilateral, so 25 is indeed the minimum ratio.